CSCI 2100 Tutorial 9

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Outline

- A review on the binary heap
- Regular exercise 8 problem 4
- Special exercise 8 problem 4

Binary Heap (Review)

Let *S* be a set of n integers. A binary heap on *S* is a binary tree *T* satisfying:

1. *T* is a complete binary tree.

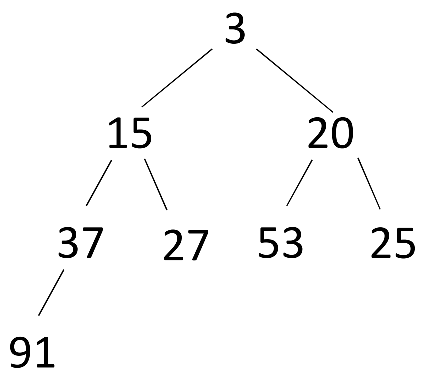
2. Every node *u* in *T* stores a distinct integer in S, called the key of u.

3. If *u* is an internal node, the key of *u* is smaller than those of its child nodes.

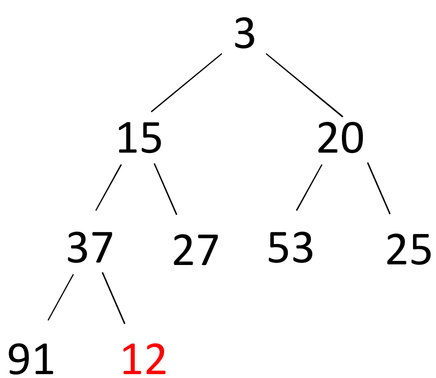
The third property may be violated after insertion and delete-min.

Heap Property Violation

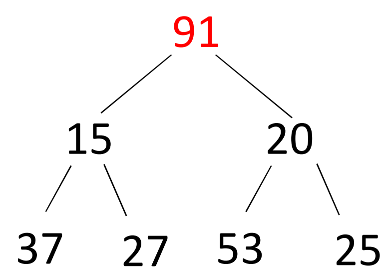
Original:



After insertion:



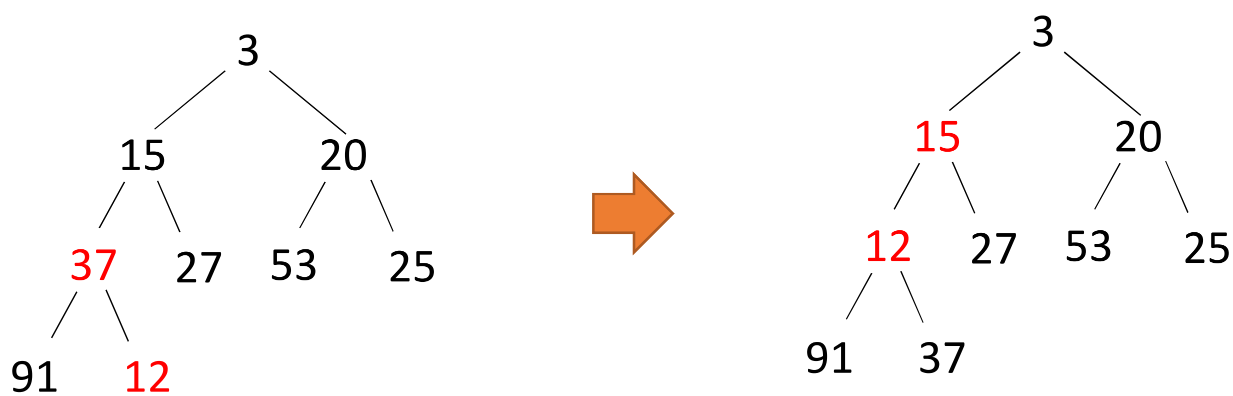
After delete-min:

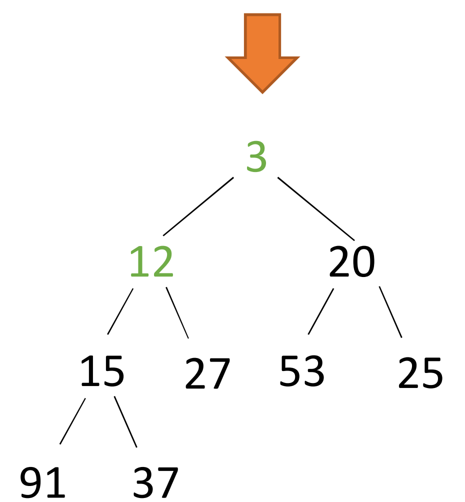


Restoring the Heap Property After Insertion

Swap up:
If node *u* has a smaller key than its parent *p*, swap the keys of *u* and *p*. Set *u* to *p*, and repeat until there is no violation.

Swap Up



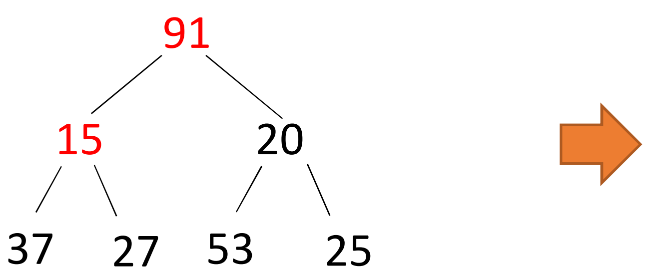


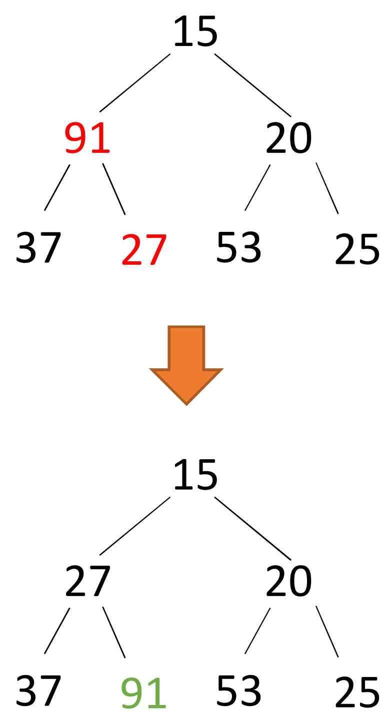
Swap up at most $O(\log n)$ times to restore the heap property.

Restoring the Heap Property After Delete-min

Swap down:
Let *v* be the child of node *u* with a smaller key. If the key of *u is* larger than the key of *v*, swap the keys of *u* and *v.* Set *u to v,* and repeat until there is no violation.

Swap Down





Swap down at most $O(\log n)$ times to restore the heap property.

Regular Exercise 8 Problem 4

Problem:

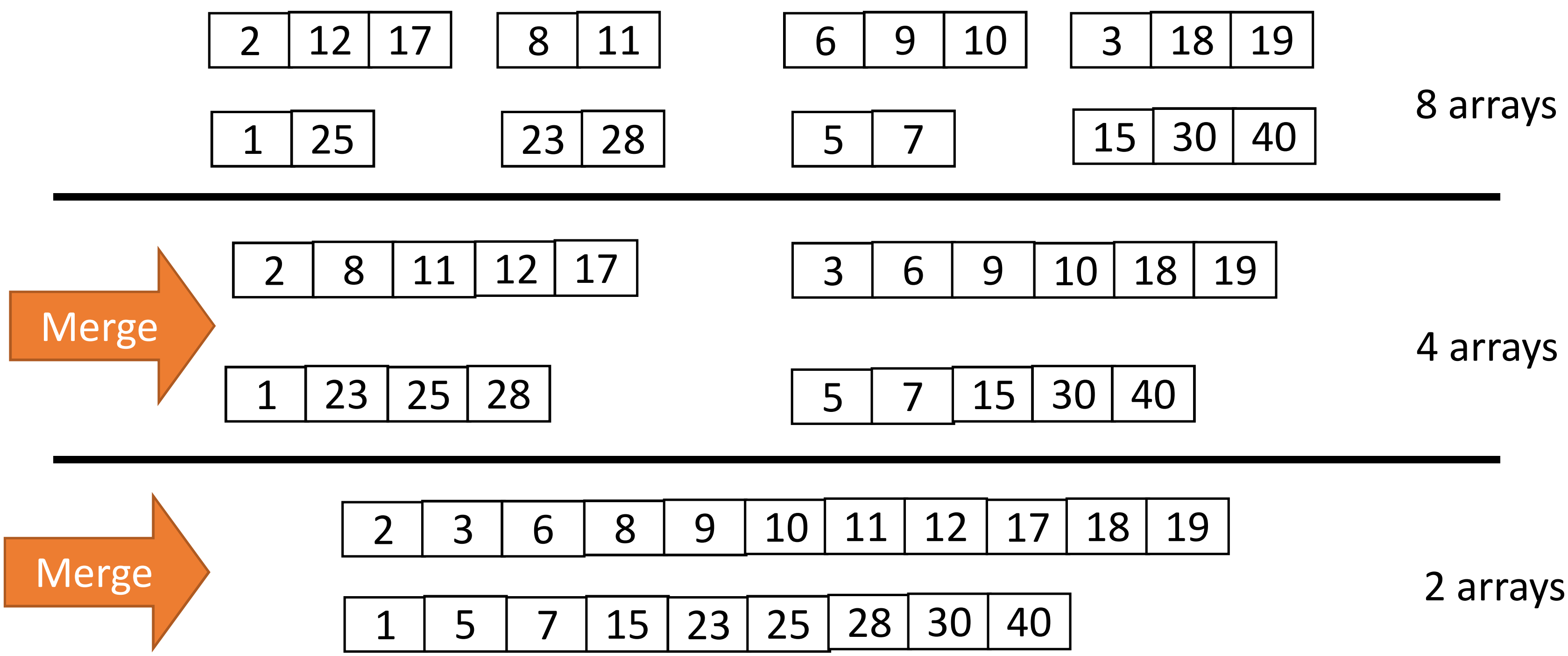
Suppose that we have *k* sorted arrays (in ascending order) $A_1, A_2, \ldots, A_k$ of integers. Let *n* be the total number of integers in those arrays.

Describe an algorithm to produce an array that sorts all the *n* integers in ascending order in $O(n \log k)$ time.

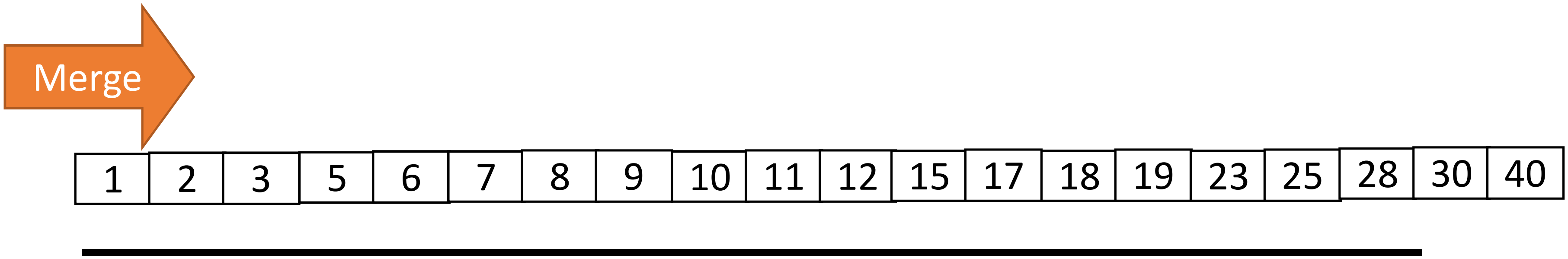
Solution 1: Merge Operation

- Input

k = 8, n = 20



Solution 1: Merge Operation



Need $O(\log k)$ passes. Each pass takes $O(n)$ time on n integers (the cost of merging is proportional to the number of elements involved).

Therefore, the total time complexity is $O(n \log k)$.

Solution 2: Binary Heap

- Input:

  k = 3, n = 15

2	15	30	40	47

5	8	11	12

9	14	21	26	27	37

- Output

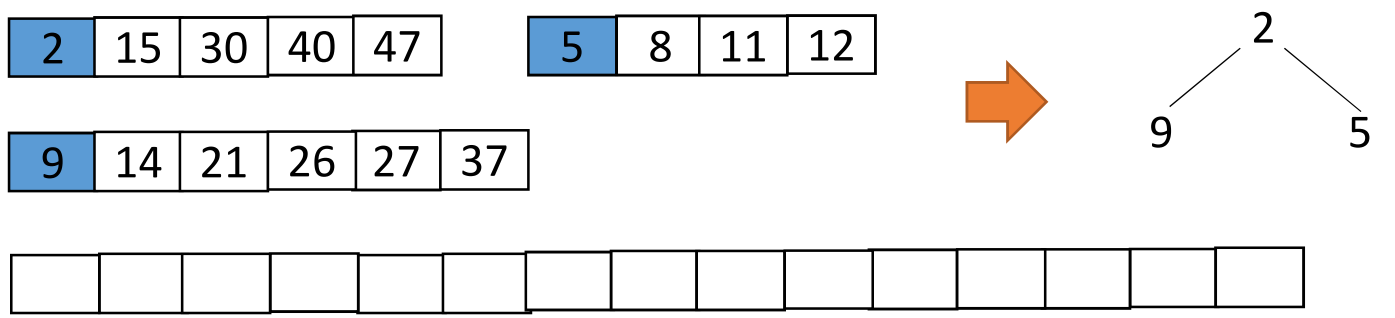
2	5	8	9	11	12	14	15	21	26	27	30	37	40	47

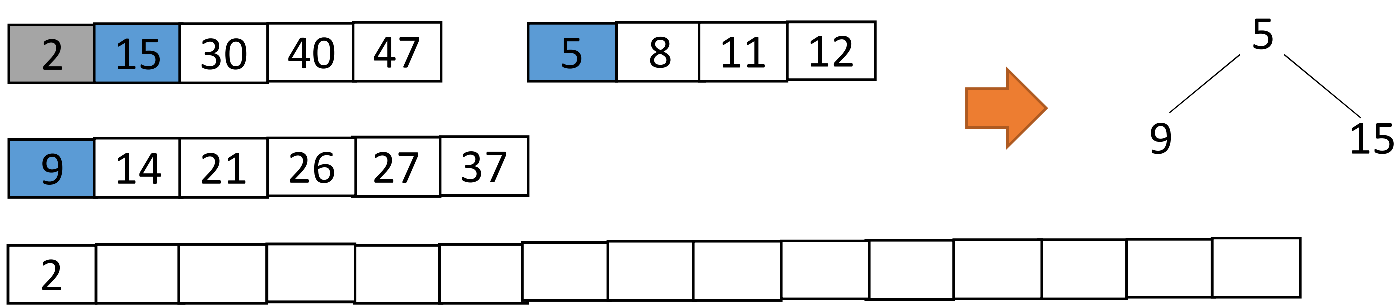
Solution 2: Binary Heap

Ideas:

- A binary heap of size k can perform delete-min and insertion in $O(log\ k)$ time.
- Perform a delete-min to obtain the smallest integer that has not been output.
- After delete-min, insert a new integer into the heap from the integer's origin array.

Solution 2: Binary Heap





Solution: Binary Heap

Initialization cost:

creating the output array: $O(n)$

Processing cost:

n insertions: $O(n\log k)$ n delete-min: $O(n\log k)$

Total time complexity:

$O(n\log k)$

Special Exercise 8 Problem 4

Problem:

Let *S* be a dynamic set of integers. At the beginning, *S* is empty. Then, new integers are added to it one by one, but never deleted. Let *k* be a fixed integer. Describe an algorithm which achieves the following guarantees:

- Space consumption $O(k)$.
- Insert(*e*): Insert a new element *e* into *S* in $O(log\ k)$ time.
- Report-top-*k*: Report the *k* largest integers in *S* in O(*k*) time.

Special Exercise 8 Problem 4

Example:

Suppose that $k = 3$, and the sequence of integers inserted is 83, 21, 66, 5, 24, 76, 92, 33, 43,…

The 3 largest integers are 83, 66, 24 after the insertion of 24, they become 83, 66, 76 after the insertion of 76, and so on.

Solution

Intuition:

- A heap *H* of size *k* takes $O(k)$ space.
- *H* performs insertion and delete-min in $O(log\ k)$

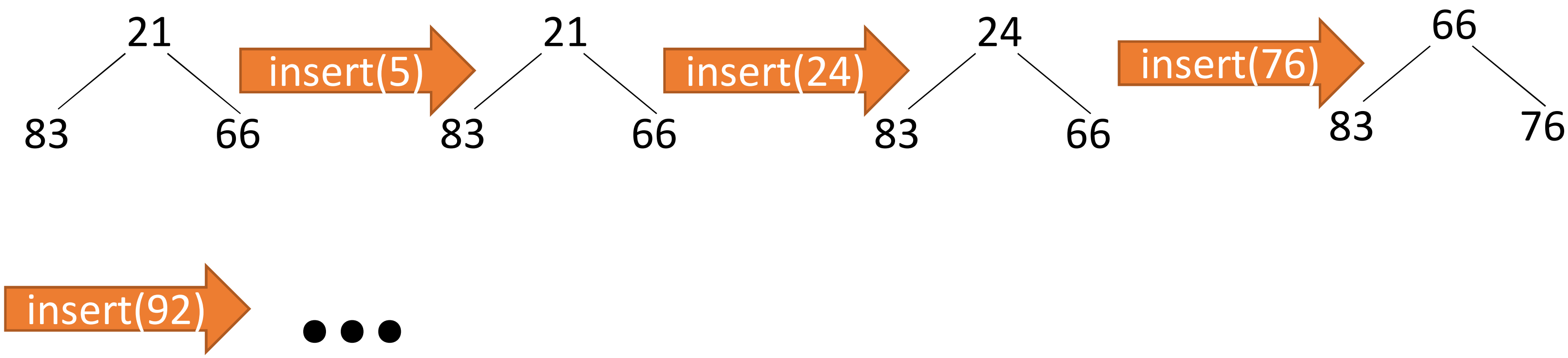
time.

- The root *r* of *H* stores the minimal integer in *H.*
- Make sure that *H* always contains the *k* largest integers. If the incoming integer *m* is larger than the minimal integer stored in *H.* We perform delete-min and insert*(m)*. Otherwise, we do nothing.

Solution

- Input:

83, 21, 66, 5, 24, 76, 92, 33, 43, …, and k=3



Solution

Maintain a binary heap *H* with *k* integers.

1. Insert first k integers into *H*. Each insertion takes $O(log\, k)$ time.
2. For a newly added integer *e* from the sequence, compare it with the integer e_r stored at the root *r* of *H*:

   (1) If *e* > e_r, perform delete-min and insert*(e)*, which take $O(log\, k)$ time in total.

   (2) Otherwise, ignore *e.*

Solution

Report-top-*k:*

Report all integers in *H* by traversing the heap.

A challenging problem for you

- For this problem, we can actually achieve
  - O(k) space
  - $O(1)$ amortized insertion time
  - O(k) top-k report time.

- Hint: k-selection.