More on Hashing

CSCI2100 Tutorial 7

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Review on Hash Table

- Given a set of n integers S in $[1, U]$
- Main idea: divide S into a number m of disjoint subsets
- Guarantees
  - Space consumption: $O(n + m)$
  - Preprocessing cost: $O(n + m)$
  - Query cost: $O(1 + n/m)$ in expectation

Review on Hash Table

- Given a set of n integers S in $[1, U]$
- Main idea: divide S into a number m of disjoint subsets
- Set $m = \Theta(n)$
- Guarantees
  - Space consumption: $O(n)$
  - Preprocessing cost: $O(n)$
  - Query cost: $O(1)$ in expectation

Review on Hash Table

- Divide S into a number m of disjoint subsets:
  - Choose a function h from $[1, U]$ to $[1, m]$
  - For each $i \in [1, m]$, create an empty linked list L_i
  - For each $x \in S$:
    - Compute $h(x)$
    - Insert x into $L_{h(x)}$

- Important:
  - Choose a good hash function h

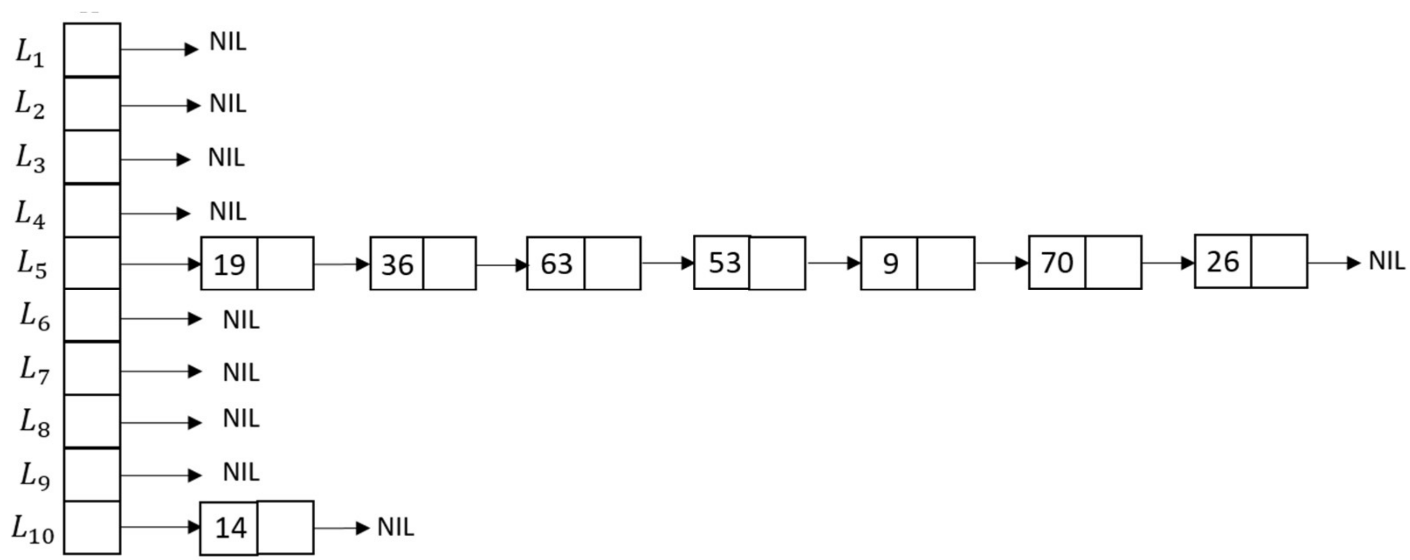
Review on Hash Table

- Construct a universal family
  - Pick a prime number p such that $p \geq m$ and $p \geq U$
  - Choose an integer α from $[1, p-1]$ uniformly at random
  - Choose an integer β from $[0, p-1]$ uniformly at random
  - Define a hash function:

$$h(k) = 1 + \big((\alpha k + \beta) \bmod p\big) \bmod m$$

Example

- Let $S = \{19,36,63,53,14,9,70,26\}$
- We choose $m = 10, p = 71$, suppose that α and β are randomly chosen to be 3 and 7, respectively
- $h(k) = 1 + (((3k + 7) \bmod 71) \bmod 10)$



Hash Table

- Let H be the universal family defined in the previous slides
- Given a function $h \in H$ and an integer q$\in [1, U]$:
  - Define $\text{cost}(h, q) = |\{x \in S \mid h(x) = h(q)\}|$

query value

	1	2	…	U	Max
h_1	$\text{cost}(h_1, 1)$	$\text{cost}(h_1, 2)$	…	$\text{cost}(h_1, U)$	$O(n)$
h_2	$\text{cost}(h_2, 1)$	$\text{cost}(h_2, 2)$	…	$\text{cost}(h_2, U)$	$O(n)$
…	…	….	…	….	$O(n)$
$h_{\|H\|}$	$\text{cost}(h_{\|H\|}, 1)$	$\text{cost}(h_{\|H\|}, 2)$	…	$\text{cost}(h_{\|H\|}, U)$	$O(n)$
Average	$O(1)$	$O(1)$	$O(1)$	$O(1)$	

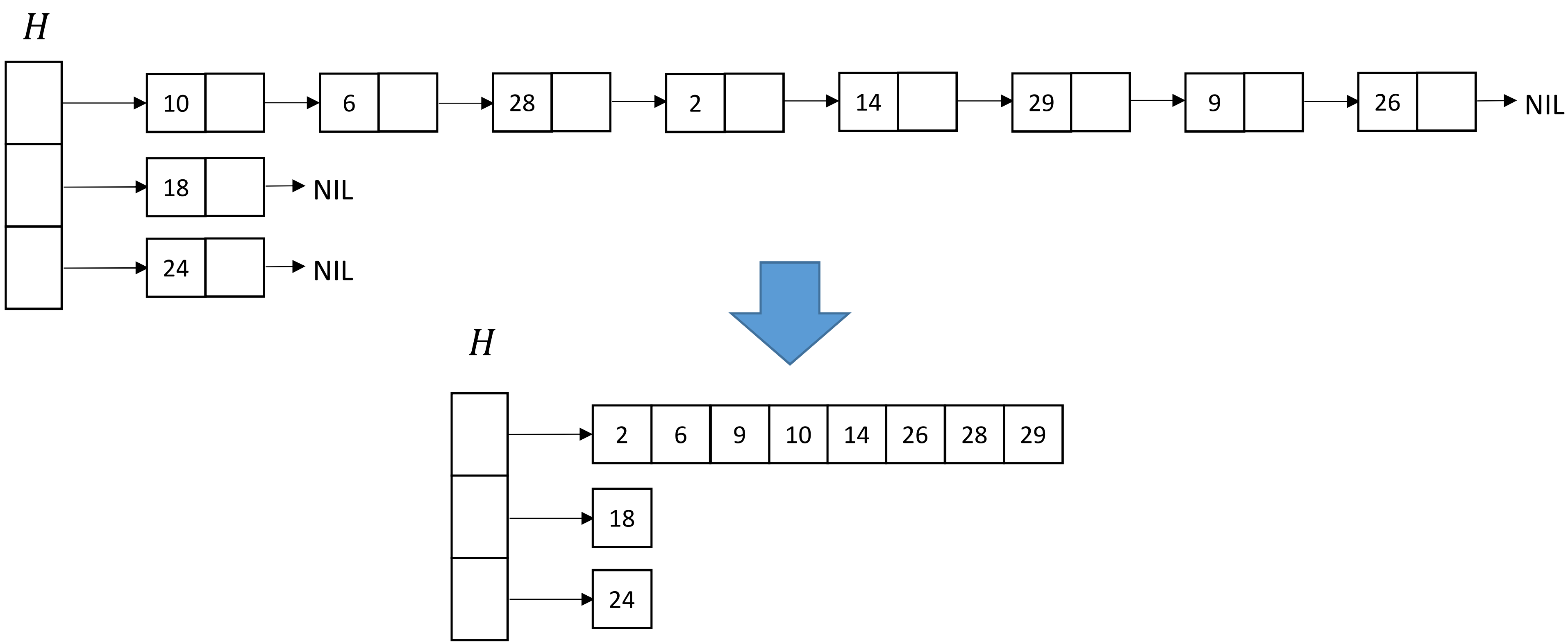
Hash Table

- Worst-case expected query cost: $O(1)$
  - Pick a hash function from a universal family
- Worst-case query cost: $O(n)$
  - All elements are hashed into the same value

- Question:
  - Can we improve the worst-case query cost?

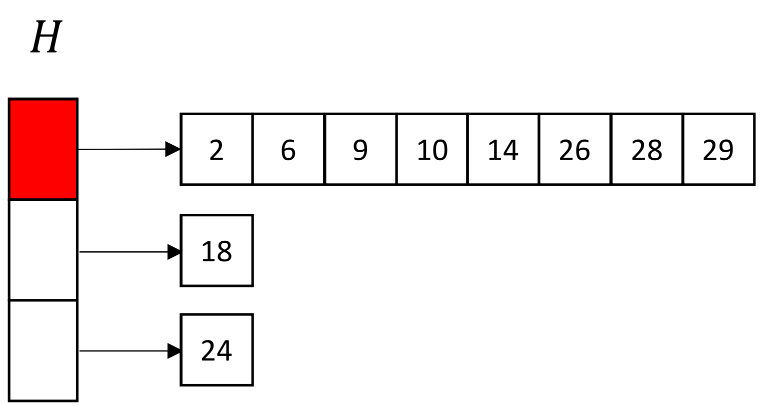
Hash Table

- Replace linked lists with arrays
- Sort the arrays, cost $O(n \log n)$ for preprocessing



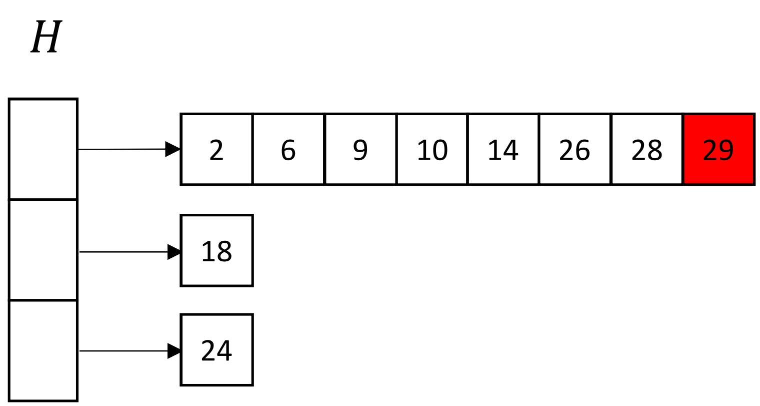
Hash Table

- Query: whether 29 exists
- Step 1:
  - Access the hash table to obtain the address of corresponding array
    - $O(1)$ time



Hash Table

- Query: whether 29 exists
- Step 2:
  - Perform binary search on the array to find the target
    - $O(\log n)$ time
- Overall worst-case complexity: $O(\log n)$



Hash Table

- This method retains the $O(1)$ worst-case expected query time.
- Proof:
  - Suppose we look up an integer q
  - Define random variable $X_{h(q)}$ to be the length of array that corresponds to the hash value $h(q)$
  - Expected query time:
    - $\mathrm{E}\left[\log_2 X_{h(q)}\right] = \sum_{l=1}^{n} \log_2 l \Pr\left(X_{h(q)} = l\right)$
    - $\leq \sum_{l=1}^{n} l \Pr\left(X_{h(q)} = l\right)$
    - $= \mathrm{E}\left[X_{h(q)}\right]$
    - $= O(1)$

The Two-Sum Problem (revisited)

- Problem Input:
  - A set S of unsorted n distinct integers
  - The value n has been placed in Register 1
  - A positive integer v has been placed in Register 2
- Goal:
  - Determine whether if there exist two different integers x and y in S such that $x + y = v$
- For example:
  - Find a pair whose sum is 20

11	3	17	7	2	13

Solution 1: Binary Search the Answer

- Goal: Find $a\ pair\ (x, y)$ such that $x + y = v$
- Observe that given x, $y = v - x$, is determined
- Solution:
  - Sort S
  - For each x in S:
    - set y as $v - x$
    - Use binary search to see if y exists in the sequence
- Time complexity: $O(n \log n)$

Solution 2: Using the Hash Table

- Step 1 and 2:
  - Choose a hash function h and create an empty hash table H
  - Insert each x in S into $L_{h(x)}$

- Step 3:
  - For each x in S:
    - Set y as $v - x$
    - Check if y is in the hash table; if it is, return yes
  - Return no

Time Complexity

- Step 1 and 2: $O(n)$
- Step 3:
  - Let X_i be the query time for the i-th integer in S
  - We know $E[X_i] = O(1)$
  - Define $X = \sum_{\mathrm{i}} X_i$
  - The worst-case expected cost of step 3:
    - $E[\mathrm{X}] = \sum_i E[\mathrm{X_i}] = \mathrm{O(n)}$
- Overall: $O(n)$ in expectation