# Exercises on the Growth of Functions

## CSCI2100 Tutorial 2

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Adapted from the slides of the previous offerings of the course

## Introduction

Recall the definition of $f(n) = O(g(n))$:

> $f(n) = O(g(n))$, if there exist two positive constants $c_1$ and $c_2$ such that $f(n) \leq c_1 \cdot g(n)$ holds for all $n \geq c_2$.

Last week, we have learned two different ways to decide whether one function $f(n) = O(g(n))$:

- Finding appropriate "constants $c_1, c_2$" to prove existence.
- if $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists and is less or equals to some constant $c \geq 0$, then $f(n) = O(g(n))$.

In this tutorial, we will apply both methods through some exercises.

## Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

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Proof of $f(n) = O(g(n))$

Direction 1: Constant Finding

$f(n) = O(g(n))$, if there exist two positive constants $c_1$ and $c_2$ such that $f(n) \leq c_1 \cdot g(n)$ holds for all $n \geq c_2$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Proof of $f(n) = O(g(n))$

Direction 1: Constant Finding

Our mission is to find $c_1, c_2$ to make $f(n) \leq c_1 \cdot g(n)$ hold for all $n \geq c_2$. Remember: we do not need to find the smallest $c_1, c_2$; instead, it suffices to obtain any $c_1, c_2$ that can do the job. Indeed, we will often go for some "easy" selections that can simplify derivation.

## Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

### Direction 1: Constant Finding

(try $c_1 = 5$)

$$\begin{aligned} & f(n) \le c_1 \cdot g(n) \\ \Leftrightarrow \quad & 10n + 5 \le c_1 \cdot n^2 \\ \Leftrightarrow \quad & 5(2n+1) \le 5 \cdot n^2 \\ \Leftrightarrow \quad & 2n + 1 \le n^2 \\ \Leftrightarrow \quad & 2 \le (n-1)^2 \\ \Leftarrow \quad & 3 \le n \end{aligned}$$

Hence, it suffices to set $c_2 = 3$. So there exist positive constants $c_1, c_2$ namely $c_1 = 5, c_2 = 3$ such that $f(n) \le c_1 \cdot g(n)$ holds for all $n \ge c_2$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Proof of $f(n) = O(g(n))$

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

$$\lim_{n\to\infty} \frac{10n+5}{n^2} = \lim_{n\to\infty} \frac{10+5/n}{n} = 0.$$

Hence, $f(n) = O(g(n))$.

## Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

### Proof of $g(n) \neq O(f(n))$

### Prove by contradiction

Let us prove this by contradiction. Suppose, on the contrary, that $g(n) = O(f(n))$. This means the existence of constants $c_1, c_2$ such that, we have for all $n \geq c_2$

$$
\begin{aligned}
& n^2 \leq c_1 \cdot (10n + 5) \\
\Rightarrow \quad & n^2 \leq c_1 \cdot 20n \\
\Leftrightarrow \quad & n \leq 20c_1
\end{aligned}
$$

which cannot always hold for all $n \geq c_2$. This completes the proof.

Exercise 2

Let $f(n) = 5\log_2 n$ and $g(n) = \sqrt{n}$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

## Proof of $f(n) = O(g(n))$

### Direction 1: Constant Finding

Setting $c_1 = 5$, we want:

$$\begin{aligned} & 5\log_2 n \le 5 \cdot \sqrt{n} \\ \Leftrightarrow \quad & \log_2 n \le \sqrt{n} \end{aligned}$$

Hence, it suffices to set $c_2 = 64$. So there exist positive constants $c_1, c_2$ namely $c_1 = 5, c_2 = 64$ such that $f(n) \le c_1 \cdot g(n)$ holds for all $n \ge c_2$.

Proof of $f(n) = O(g(n))$

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{5\log_2 n}{\sqrt{n}} = 0.$$

Thus, we have $f(n) = O(g(n))$.

Proof of $g(n) \neq O(f(n))$

Prove by Contradiction

We prove this by contradiction. Suppose that $g(n) = O(f(n))$. It implies that there exist constants $c_1, c_2$ such that for all $n \geq c_2$, we have

$$\begin{aligned} & \sqrt{n} \leq c_1 \cdot 5 \cdot \log_2 n \\ \Leftrightarrow \quad & \frac{\sqrt{n}}{\log_2 n} \leq 5c_1 \end{aligned}$$

which cannot always hold for all $n \geq c_2$. This completes the proof.

Exercise 3

Given that $f(n) = O(g(n))$ where $f(n), g(n) \geq 0$, prove $\sqrt{f(n)} = O(\sqrt{g(n)})$.

Since $f(n) = O(g(n))$ implies the existence of constants $c_1$ and $c_2$ such that $f(n) \leq c_1 \cdot g(n)$ holds for all $n \geq c_2$.

Thus:

$$\sqrt{f(n)} \leq \sqrt{c_1 \cdot g(n)} = \sqrt{c_1} \cdot \sqrt{g(n)}$$

holds for all $n \geq c_2$.

Therefore, there exist positive constants $c_1^{'}, c_2^{'}$ namely $c_1^{'} = \sqrt{c_1}, c_2^{'} = c_2$ such that $\sqrt{f(n)} \leq c_1 \cdot \sqrt{g(n)}$ holds for all $n \geq c_2^{'}$.

Exercise 4

Consider functions of $n$: $f_1(n)$, $f_2(n)$, $g_1(n)$ and $g_2(n)$ such that:

$$f_1(n) = O(g_1(n)) \text{ and } f_2(n) = O(g_2(n))$$

Prove $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.

Since $f_1(n) = O(g_1(n))$, there exist constants $c_1$ and $c_2$ such that $f_1(n) \leq c_1 \cdot g_1(n)$ holds for all $n \geq c_2$.

Similarly, $f_2(n) = O(g_2(n))$ implies the existence of constants $c_1'$ and $c_2'$ such that $f_2(n) \leq c_1' \cdot g_2(n)$ holds for all $n \geq c_2'$.

Thus:

$$f_1(n) + f_2(n) \leq c_1 \cdot g_1(n) + c_1' \cdot g_2(n) \leq \max\{c_1, c_1'\} \cdot (g_1(n) + g_2(n))$$

for all $n \geq \max\{c_2, c_2'\}$.

Therefore, $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.