Comparison Lower Bound of Sorting (Slides for ESTR2102)

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We already know that n elements can be sorted in $O(n \log n)$ time. This lecture will prove that the time complexity is optimal for comparison-based algorithms. In other words, every such algorithm must incur $\Omega(n \log n)$ time on at least one input.

There are $n!$ different ways to permute the n elements in the input array A.

Example

For $n = 3$, 6 permutations:

$$A[1], A[2], A[3]$$
$$A[1], A[3], A[2]$$
$$A[2], A[1], A[3]$$
$$A[2], A[3], A[1]$$
$$A[3], A[1], A[2]$$
$$A[3], A[2], A[1]$$

The goal of sorting is essentially to decide which of the $n!$ permutations is the final sorted order.

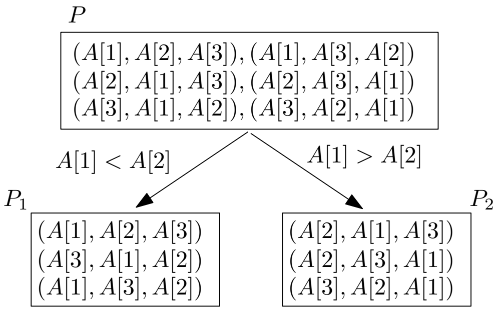
Comparison-Based Algorithm

Formally, such an algorithm works by continuously shrinking a pool P of possible permutations.

- At the beginning, P contains all the $n!$ permutations.
- Every comparison allows the algorithm to discard all those permutations in P that are inconsistent with the comparison's result.
- Eventually, P has only 1 permutation left, which is thus the final sorted order.

In other words, at any moment, all the permutations that remain in P are possible results. The algorithm cannot terminate as long as $|P| \geq 2$.

Shrinking the Pool: An Example



In general, each comparison allows us to shrink P to either P_1 or P_2.

Comparison-Based Algorithm: The Framework

Framework

1. $P \leftarrow$ all the $n!$ permutations of A
2. **while** $|P| > 1$
3. make a comparison between elements e_1 and e_2
4. **if** $e_1 < e_2$ **then**
5. $P \leftarrow P_1$, where P_1 is the set of permutations in P consistent with $e_1 < e_2$
6. **else**
7. $P \leftarrow P_2$, where P_2 is the set of permutations in P consistent with $e_1 > e_2$
8. **return** the permutation in P

Various algorithms differ in how they implement Step 3.

A Worst-Case Lower Bound

- Note that one of P_1 and P_2 contains at least half of the permutations in P (i.e., either $|P_1| \geq |P|/2$ or $|P_2| \geq |P|/2$).
- The worst case happens when P always shrinks to the larger set between P_1 and P_2.
- In this case, the size of P shrinks by at most half after each comparison.
- Hence, the number of comparisons required before $|P|$ decreases to 1 is $\log_2(n!)$.

The next slide shows $\log_2(n!) = \Omega(n \log n)$.

A Worst-Case Lower Bound

$$
\begin{aligned}
\log_2(n!) &= \sum_{i=1}^{n} \log_2 i \\
&\geq \sum_{i=n/2}^{n} \log_2 i \\
&\geq (n/2)\log_2(n/2) \\
&= \Omega(n \log n).
\end{aligned}
$$

We now conclude that any comparison-based algorithm must incur $\Omega(n \log n)$ time sorting n elements in the worst case.