Charging Arguments
[Notes for ESTR2102]

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Recall

In general, if a data structure can process any n operations in $f(n)$ time, we say that it guarantees an **amortized cost** of $\frac{f(n)}{n}$ per operation.

Today, we will learn a **charging argument** technique to prove amortized costs.

Ideas behind a Charging Argument

Consider n operations on a data structure. The i-th ($1 \leq i \leq n$) operation incurs cost C_i. Our goal is to prove:

$$\sum_{i=1}^{n} C_i \leq f(n). \tag{1}$$

Suppose that we can **assign** a "fake" cost $\overline{C_i} \leq \frac{f(n)}{n}$ to the i-th operation such that

$$\sum_{i=1}^{n} C_i \leq \sum_{i=1}^{n} \overline{C_i}. \tag{2}$$

(1) will then follow from (2).

Recall: the Dynamic Array Problem

Let S be a collection of integers (not necessarily distinct). S is empty in the beginning. Integers are then added to S one by one with **insertions**.

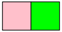
Let n be the number of elements in S currently. We want to maintain an array A satisfying:

1. A has length $O(n)$.
2. For each $i \in [1, n]$, $A[i] = x$ if x is the i-th integer added to S.

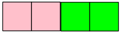
The above requirements need to be satisfied after every insertion.

Recall: The Expansion Algorithm

$n = 1$

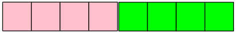


$n = 2$



$n = 3$

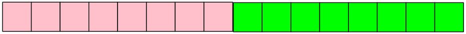
$n = 4$



$n = 5$

...

$n = 8$



Charging Argument

Earlier, we proved that each insertion has amortized cost $O(1)$. Next, we give an alternative analysis for proving the same.

Our algorithm ensures an invariant:

> After an expansion, the new array has size $2n$, namely, there are n **empty positions**.

Charging Argument

Let C_i be the actual cost of the i-th insertion.

We will assign an **amortized cost** $\overline{C_i}$ to the i-th insertion.

Charging Argument

For the n-th operation, first set $\overline{C_n} = O(1)$.

If the array does not expand, done.

An array expansion takes at most cn time for some constant c.

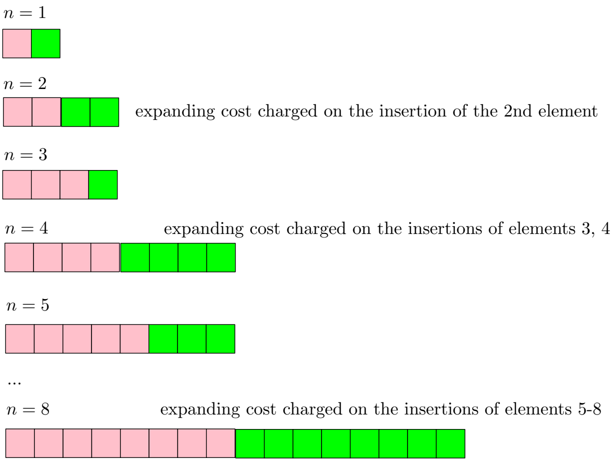
$\Rightarrow$ The previous expansion happened when S had $n/2$ elements.

$\Rightarrow$ $n/2$ empty positions in the previous array.

$\Rightarrow$ $n/2$ insertions have taken place since the previous expansion.

$\Rightarrow$ **Charge the cn cost over those $n/2$ insertions**: for each of those insertions, add $\frac{cn}{n/2} = 2c = O(1)$ to its amortized cost.

Example



Each insertion is charged at most once.

Charging Argument

Convince yourself:

$$\sum_{i=1}^{n} C_i \le \sum_{i=1}^{n} \overline{C_i}$$

and

$$\overline{C_i} = O(1).$$

Therefore, the total cost of all the n operations is $O(n)$.

The Stack-with-Array Problem

Let S be a collection of integers (not necessarily distinct). We want to support:

- push(e): add an integer e into S.
- pop: remove the **most recently** inserted integer from S.

At any moment, let m be the number of elements in S. We want to store all the elements of S in an array A satisfying:

1. A has length $O(m)$
2. $A[1]$ is the **least recently** inserted element, $A[2]$ the **second least recently** inserted, ..., $A[m]$ the most recently inserted.

We will denote by n the number of operations processed so far.

The Stack-with-Array Problem

We will give an algorithm for maintaining such an array by handling n operations in $O(n)$ time, namely, each operation is processed in $O(1)$ amortized time.

The Stack-with-Array Problem

1. A is **full** if all its cells are filled.
2. A is **sparse** if at most $1/4$ of its cells are filled.

We will enforce an invariant:

> At creation, an array is **half full** (i.e., half of its cells are filled).

Push

Carry out push(e) in the same way we perform an insertion in the dynamic array problem.

Pop

Perform pop as follows:

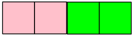
- Return the last element of A and decrease n by 1. If A is sparse, then:
  - Initialize an array A' of length $2n$.
  - Copy all the n elements of A over to A'.
  - Destroy A and replace it with A'.

Example

11 pushes followed by 9 pops on an initially empty stack:

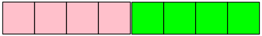
$n = 1$, push

$n = 2$, push



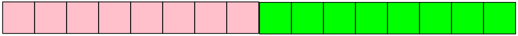
...

$n = 4$, push



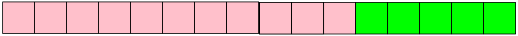
...

$n = 8$, push



...

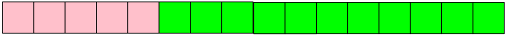
$n = 11$, push



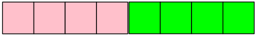
Example

...

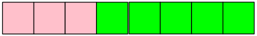
$n = 17$ pop



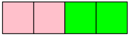
$n = 18$, pop



$n = 19$, pop



$n = 20$, pop



Think: how to prove that each operation incurs only $O(1)$ amortized cost?