Breaking Value Symmetries in Matrix Models using Channeling Constraints

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ABSTRACT

Multi-aspect Assignment Problems (MAPs) can be naturally formulated into various matrix models of Constraint Satisfaction Problems (CSPs), which can contain both variable and value symmetries, using different viewpoints. While variable symmetry breaking constraints can be expressed relatively easily and executed efficiently by enforcing lexicographic ordering, value symmetry breaking constraints are difficult to formulate. We show when value symmetries in one viewpoint correspond to variable symmetries in another, and when symmetry breaking constraints in two viewpoints are consistent. Our results allow tackling value symmetries efficiently using additional viewpoints and channeling constraints. Experiments on the social golfer problem and a variant of the quasigroup existence problem confirm the benefits of our proposal against conventional methods.

Keywords

CSP, symmetry breaking

1. INTRODUCTION

The social golfer problem (SGP), "prob010" in CSPLib,[1] is to find a $\mathcal{W}$-week schedule of $\mathcal{G}$ groups, each containing $\mathcal{S}$ golfers, such that no two golfers can play together more than once. There are totally $\mathcal{N} = \mathcal{G} \times \mathcal{S}$ golfers. We denote an instance of the problem as $(\mathcal{G}, \mathcal{S}, \mathcal{W})$. There are three *aspects* in the problem, corresponding to the sets of golfers, weeks, and groups respectively. Solving the problem is to find a set of tuples of the form $(aGolfer, aWeek, aGroup)$ that satisfies the problem requirements. The SGP is a *Multi-aspect Assignment Problem* (MAP).[2] A MAP consists of n aspects, each of which corresponds to a set of objects of the problem. Without loss of generality, we define the set of objects of the i-th aspect as $Obj(i) = \{1, \ldots, k_i\}$, where k_i is the number of objects in aspect i. For example, we can use $Obj(1)$, $Obj(2)$, and $Obj(3)$ to denote the set of all golfers, weeks, and groups respectively in the SGP. Solving a MAP is to find a *solution set* of tuples $S \subseteq Obj(1) \times \ldots \times Obj(n)$ that satisfies the problem constraints. For example, a tuple $(1, 2, 3)$ in a solution set of the SGP means that golfer 1 plays in group 3 in week 2. Many real life problems, such as timetabling and resource allocation, are MAPs, which can readily be formulated into *matrix models* [5] of *Constraint Satisfaction Problems* (CSPs) [10]. In a matrix model, the CSP variables can be indexed and organized into matrices.

There are two common types of symmetries in CSPs, namely variable and value symmetries. We observe that, in matrix models, it is more difficult to express symmetry breaking constraints for value symmetries than those for variable symmetries. Our goal is to tackle value symmetries in matrix models using multiple viewpoints and channeling constraints [1]. Flener *et al.* [5] suggest that it is possible to transform an $(n-1)$-dimensional matrix with variable and value symmetries into an n-dimensional matrix that only contains variable symmetries. Symmetry breaking constraints are then expressed in the n-dimensional matrix to break all kinds of symmetries of the problem. We formally describe this idea by theoretically showing that value symmetries in a matrix model always correspond to variable symmetries in the 0/1 viewpoint. We describe a general method to derive $n+1$ viewpoints for a MAP with n aspects. We then generalize the idea to characterize the conditions of when value symmetries in one viewpoint correspond to variable symmetries in non-0/1 viewpoints. We also address the consistency issue for symmetry breaking constraints in multiple viewpoints. Such results enable us to break value symmetries in one viewpoint using variable symmetry breaking constraints in another. We demonstrate the feasibility of our proposal using both integer and set models of the SGP, as well as a variant of the quasigroup existence problem. The models contain both integer and set value symmetries, and experimental results confirm the efficiency of our approach in terms of number of fails and execution time.

2. BACKGROUND

A *CSP viewpoint* (or simply *viewpoint*) is a pair $V = (X, D_X)$, where $X = \{x_1, \ldots, x_n\}$ is a set of *variables*, and D_X is a function that maps each $x \in X$ to its associated *do-*

[1] Available at `http://www.csplib.org/`.

[2] A Multi-aspect Assignment Problem is different from a Multidimensional Assignment Problem [11], which is an optimization problem subject to some constraints in particular forms.

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main $D_X(x)$, giving the set of possible values for x. There are two common classes of variables in CSPs. An *integer variable* [8] x has an integer domain, i.e., $D_X(x)$ is an integer set. A *set variable* [8] x has a set domain, i.e., each element in the domain is a set. In most implementations, the domain of a set variable x is represented by two sets. The *possible set* $PS(x)$ contains elements that belong to at least one of the possible values of the variable. The *required set* $RS(x)$ contains elements that belong to all the possible values of the variable. For ease of description, we abuse terminology by saying that the possible set $PS(x)$ of an integer variable x is $D_X(x)$.

A viewpoint $V = (X, D_X)$ defines the possible decisions for variables in X. A *decision* $x \mapsto b$ in V means that variable $x \in X$ is mapped to the value $b \in PS(x)$. It has different meanings depending on the class of variable x. If x is an integer variable, $x \mapsto b$ simply means x is assigned the value b, i.e., $x = b$. If x is a set variable, $x \mapsto b$ means that the value b is added to the required set of x, i.e., $b \in x$. Note that decisions are different from assignments in that multiple decisions are allowed for a set variable, while multiple assignments are not allowed for any variable. A *compound decision* is a set of decisions $\{x_{i_1} \mapsto a_1, \ldots, x_{i_k} \mapsto a_k\}$, where $\{x_{i_1}, \ldots, x_{i_k}\} \subseteq X$. Note the requirement that no integer variables can occur more than once in a compound decision. The *scope* of a compound decision is a variable set indicating the assigned variables. For example, for integer variable x and set variables y and z, the compound decision $\{x \mapsto 1, y \mapsto 1, y \mapsto 2\}$ with scope $\{x, y, z\}$ means $x = 1$, $y = \{1, 2\}$, and $z = \emptyset$. We overload the $\mapsto$ operator to represent compound decisions such that $\langle x_{i_1}, \ldots, x_{i_k}\rangle \mapsto \langle a_1, \ldots, a_k\rangle$ means $\{x_{i_j} \mapsto a_j \mid 1 \le j \le k\}$.

A *constraint* places restrictions on a subset of variables in V, limiting the combination of values that these variables can take. A *CSP model* M (or simply *model*) of a problem P is a pair (V, C), where V is a viewpoint of P and C is a set of constraints in V for P. A *solution* of (V, C) is a compound decision in V with scope X satisfying all the constraints in C. The set of all solutions of a CSP M is denoted as *sol*(M).

3. VARIABLE AND VALUE SYMMETRIES

In this section, we define two types of symmetries, namely variable and value symmetries. We first describe the symmetries of the SGP [4]: (1) players can be permuted among the $\mathcal{N}!$ combinations, (2) weeks of schedule can be exchanged, and (3) groups can be exchanged inside weeks.

One way to model the problem into a CSP uses the viewpoint $V_G = (G, D_G)$ which contains a variable $g_{i,k}$ for each golfer i in week k with $1 \le i \le \mathcal{N}$ and $1 \le k \le \mathcal{W}$. The variable domains $D_G(g_{i,k}) = \{1, \ldots, \mathcal{G}\}$ contain the group numbers that golfer i can play in week k. A model in V_G is a matrix model, since G forms a 2-dimensional matrix of variables. Figure 1(a) gives a solution of the $(3, 2, 3)$ instance.

week \ golfer	1	2	3	4	5	6
1	1	1	2	2	3	3
2	1	2	1	3	2	3
3	1	2	2	3	3	1

(a)

group \ week	1	2	3
1	{1, 2}	{1, 3}	{1, 6}
2	{3, 4}	{2, 5}	{2, 3}
3	{5, 6}	{4, 6}	{4, 5}

(b)

golfer \ group	1	2	3
1	{1, 2, 3}	∅	∅
2	{1}	{2, 3}	∅
3	{2}	{1, 3}	∅
4	∅	{1}	{2, 3}
5	∅	{2}	{2, 3}
6	{3}	∅	{1, 2}

(c)

Figure 1: Three Equivalent Solutions of $(3, 2, 3)$ in V_G, V_P, and V_W Respectively

3.1 Variable Symmetries

A *variable symmetry* of a CSP $M = ((X, D_X), C_X)$ is a solution-preserving bijective mapping from the set of variables X to itself, $\sigma : X \to X$. We overload the σ operator to act also on a compound decision θ by defining $\sigma(\theta) = \{\sigma(x) \mapsto a \mid (x \mapsto a) \in \theta\}$. A variable symmetry σ requires that $\theta \in sol(M) \Leftrightarrow \sigma(\theta) \in sol(M)$, where $\theta \neq \sigma(\theta)$.

Symmetry (1) of the SGP is an example of variable symmetries in V_G. Consider the solution in Figure 1(a), we can exchange the variables of golfers 1 and 2 to obtain another solution with $\langle g_{1,1}, g_{1,2}, g_{1,3}\rangle \mapsto \langle 1, 2, 2\rangle$ and $\langle g_{2,1}, g_{2,2}, g_{2,3}\rangle \mapsto \langle 1, 1, 1\rangle$. Hence, we have the bijective mapping σ as the identity mapping except $\sigma(g_{1,k}) = g_{2,k}$ and $\sigma(g_{2,k}) = g_{1,k}$ for $1 \le k \le 3$. Similarly, symmetry (2) is another example of variable symmetries in V_G.

A variable symmetry σ can be broken by the lexicographic ordering constraint [7] $\langle x_1, \ldots, x_n\rangle \le_{lex} \langle \sigma(x_1), \ldots, \sigma(x_n)\rangle$ [3], where $\{x_1, \ldots, x_n\}$ is the set of variables in the CSP. Sometimes, these constraints can be simplified to contain fewer variables. An example is the row ordering and column ordering constraints for row and column symmetries [5]. For example, symmetry (1) of the SGP can be broken by the row ordering constraints $\langle g_{i,1}, \ldots, g_{i,\mathcal{W}}\rangle \le_{lex} \langle g_{i+1,1}, \ldots, g_{i+1,\mathcal{W}}\rangle$ for $1 \le i < \mathcal{N}$. Similarly, we can break symmetry (2) in V_G by the column ordering constraints $\langle g_{1,k}, \ldots, g_{\mathcal{N},k}\rangle \le_{lex} \langle g_{1,k+1}, \ldots, g_{\mathcal{N},k+1}\rangle$ for $1 \le k < \mathcal{W}$. Note that these constraints do not completely break the compositions of the row and column symmetries [5].

3.2 Value Symmetries

A *value symmetry* [5] under a subset $U \subseteq X$ of the variables of a CSP $M = ((X, D_X), C_X)$, where $PS(x) = PS(x')$ for all $x, x' \in U$, is a solution-preserving bijective mapping on the possible set of the variables in U, $\tau : PS(x) \to PS(x)$ where $x \in U$. We overload the τ operator to act also on a compound decision θ by defining $\tau(U, \theta) = \{x \mapsto \tau(a) \mid (x \mapsto a) \in \theta \wedge x \in U\} \cup \{x \mapsto a \mid (x \mapsto a) \in \theta \wedge x \notin U\}$. A value symmetry τ under U requires that $\theta \in sol(M) \Leftrightarrow \tau(U, \theta) \in sol(M)$, where $\theta \neq \tau(U, \theta)$. If U is a set of integer (*resp.* set) variables, τ is called an *integer (*resp. *set) value symmetry*.

Value symmetry is similar to value interchangeability [6]. Interchangeable values can be exchanged for a *single* variable without affecting the satisfaction of constraints, while a value symmetry can be applied to a solution to form another solution of the same CSP.

Symmetry (3) in the SGP is an example of integer value symmetries in V_G. Consider the solution in Figure 1(a). We can permute the values assigned to the set of variables $U = \{g_{1,1}, \ldots, g_{n,1}\} \subseteq G$ from 1 to 2, from 2 to 3, and from 3 to 1 to obtain another solution with $\langle g_{1,1}, \ldots, g_{6,1}\rangle \mapsto \langle 2, 2, 3, 3, 1, 1\rangle$. Thus, we have a value symmetry τ under U with $\tau(1) = 2$, $\tau(2) = 3$, and $\tau(3) = 1$.

Value symmetry breaking constraints are difficult to express in general, since we do not know beforehand which variable will be assigned which value. Value symmetries are usually handled by pre-assigning the affected variables as far as possible with some values without loss of generality. However, these pre-assignments, which must be extensible to solutions, cannot break all value symmetries in general. For example, in the SGP, without loss of generality, we can always have the pre-assignments $\langle g_{1,1}, \ldots, g_{\mathcal{S},1} \rangle \mapsto \langle 1, \ldots, 1 \rangle, \ldots, \langle g_{(\mathcal{G}-1)\mathcal{S}+1,1}, \ldots, g_{\mathcal{N},1} \rangle \mapsto \langle \mathcal{G}, \ldots, \mathcal{G} \rangle$ as well as $\langle g_{1,k}, \ldots, g_{\mathcal{S},k} \rangle \mapsto \langle 1, \ldots, \mathcal{S} \rangle$ for $k > 1$. The former breaks the value symmetries for week 1. The latter breaks the value symmetries of values 1 to $\mathcal{S}$ from week 2 and so on, but those of values $\mathcal{S}+1$ to $\mathcal{G}$ remains intact.

Symmetries of indistinguishable values is a special class of value symmetries where expressing symmetry breaking constraints is possible, albeit inefficiently. *Symmetries of a set of indistinguishable values* $\{v_1, \ldots, v_k\}$ under $U = \{x_1, \ldots, x_n\}$ implies $k! - 1$ value symmetries τ under U, where $\langle \tau(v_1), \ldots, \tau(v_k) \rangle$ is a non-identity permutation of $\langle v_1, \ldots, v_k \rangle$. Such symmetries can be broken by the symmetry breaking constraints $x_1 \neq v_j$ and $x_i = v_j \rightarrow \bigvee_{1 \leq i' < i} x_{i'} = v_{j-1}$ for $1 < i \leq n$ and $1 < j \leq k$. In symmetry (3) in V_G of the SGP, the groups $\{1, \ldots, \mathcal{G}\}$ are indistinguishable values under variables in each week. Therefore, we can express the symmetry breaking constraints $g_{1,k} \neq j$ and $g_{i,k} = j \rightarrow \bigvee_{1 \leq i' < i} g_{i',k} = j - 1$ for $1 \leq i \leq \mathcal{N}$, $1 < j \leq \mathcal{G}$, and $1 \leq k \leq \mathcal{W}$ to break the value symmetries in V_G. These if-then constraints are composed of *disjunctions* and are handled *inefficiently* in many CSP solvers.

4. BREAKING VALUE SYMMETRIES BY CHANNELING

In this section, we give results showing when a value symmetry in a CSP (V, C) corresponds to a variable symmetry in another CSP (V', C') modeling the same problem. Using these results, we can tackle value symmetries in (V, C) by expressing variable symmetry breaking constraints in V' and then connecting the two viewpoints V and V' using channeling constraints [1]. We also show how to generate *consistent* symmetry breaking constraints in V and V'. In the following, we first describe a general method to derive $n+1$ viewpoints for MAPs with n aspects.

4.1 Viewpoints for Modeling MAPs

A matrix can be multi-dimensional. We also use the array notation in addition to the subscript notation to denote the matrix variables in the following discussions for easier reading. Given a MAP with n aspects, we can always choose any $n-1$ aspects to form a matrix of variables [5] and the remaining aspect to form the variable domains. For $1 \leq s \leq n$, let $X_s = \{x_s[i_1] \cdots [i_{s-1}][i_{s+1}] \cdots [i_n] \mid \bigwedge_{1 \leq k \leq n, k \neq s} i_k \in Obj(k)\}$ be the matrix of variables using all but the s-th aspect as indices. The variable domains correspond to the objects in the s-th aspect, i.e., $PS(x_s[i_1] \cdots [i_{s-1}][i_{s+1}] \cdots [i_n]) = Obj(s)$. Integer variables can be used in X_s if the MAP only allows exactly one decision for each variable in X_s. Otherwise, set variables have to be used. Hence, we derive n different *aspect viewpoints* $V_1 = (X_1, D_{X_1}), \ldots, V_n = (X_n, D_{X_n})$ for a MAP. The subscript k in $V_k = (X_k, D_{X_k})$ denotes the aspect corresponding to the domain of V_k. The channeling constraints [1] between any two aspect viewpoints V_s and V_t $(s \neq t)$ induce a *channeling function* $f_{s,t}(x_s[i_1] \cdots [i_{s-1}][i_{s+1}] \cdots [i_n] \mapsto i_s) = x_t[i_1] \cdots [i_{t-1}][i_{t+1}] \cdots [i_n] \mapsto i_t$ from decisions in V_s to those in V_t, for $\bigwedge_{1 \leq k \leq n} i_k \in Obj(k)$. The reverse channeling function $f_{t,s}$ is simply $f_{s,t}^{-1}$.

In the SGP, V_G is an aspect viewpoint using the golfers and weeks to form the variables, and groups to form the domain. The other two aspect viewpoints of the SGP are $V_P = (P, D_P)$ and $V_W = (W, D_W)$. Viewpoint V_P (*resp.* V_W) uses the groups and weeks (*resp.* golfers and groups) to form the variables, and golfers (*resp.* weeks) to form the domain. The variables $p_{j,k} \in P$ and $w_{i,j} \in W$ are set variables with $PS(p_{j,k}) = \{1, \ldots, \mathcal{N}\}$ and $PS(w_{i,j}) = \{1, \ldots, \mathcal{W}\}$ respectively. Figures 1(a), 1(b), and 1(c) show the same solution, expressed in V_G, V_P, and V_W respectively. The channeling constraints between V_G and V_P are $g_{i,k} \mapsto j \Leftrightarrow p_{j,k} \mapsto i$, the ones between V_G and V_W are $g_{i,k} \mapsto j \Leftrightarrow w_{i,j} \mapsto k$, and the ones between V_P and V_W are $p_{j,k} \mapsto i \Leftrightarrow w_{i,j} \mapsto k$, for $1 \leq i \leq \mathcal{N}$, $1 \leq j \leq \mathcal{G}$, and $1 \leq k \leq \mathcal{W}$.

golfer \ week	1			2			3		
golfer \ group	1	2	3	1	2	3	1	2	3
1	1	0	0	1	0	0	1	0	0
2	1	0	0	0	1	0	0	1	0
3	0	1	0	1	0	0	0	1	0
4	0	1	0	0	0	1	0	0	1
5	0	0	1	0	1	0	0	0	1
6	0	0	1	0	0	1	1	0	0

(a)

golfer \ group	1	2	3
1	$\{2,3\}$	$\{1\}$	$\emptyset$
2	$\emptyset$	$\{1,2,3\}$	$\emptyset$
3	$\{2\}$	$\{3\}$	$\{1\}$
4	$\emptyset$	$\emptyset$	$\{1,2,3\}$
5	$\{1\}$	$\{2\}$	$\{3\}$
6	$\{1,3\}$	$\emptyset$	$\{2\}$

(b)

Figure 2: Two Solutions of $(3,2,3)$, in V_Z and V_W

Besides the aspect viewpoints, we can use all n aspects of a MAP to form an n-dimensional matrix of variables $Z = \{z[i_1] \cdots [i_n] \mid \bigwedge_{1 \leq k \leq n} i_k \in Obj(i_k)\}$. The variables in Z denote whether the tuple $(i_1, \ldots, i_n)$ is in a solution. Hence, $D_Z(z[i_1] \cdots [i_n]) = \{0, 1\}$, giving us the *0/1 viewpoint* $V_Z = (Z, D_Z)$. For $1 \leq s \leq n$, the channeling constraints [1] between aspect viewpoint V_s and V_Z induce a channeling function $f_{s,Z}(x_s[i_1] \cdots [i_{s-1}][i_{s+1}] \cdots [i_n] \mapsto i_s) = z[i_1] \cdots [i_n] \mapsto 1$ from decisions in V_s to *only* those of the form "$z[i_1] \cdots [i_n] \mapsto 1$" in V_Z, for $\bigwedge_{1 \leq k \leq n} i_k \in Obj(k)$ (since the channeling constraints never generate decisions of the form "$z[i_1] \cdots [i_n] \mapsto 0$"). Again, $f_{Z,s}$ is $f_{s,Z}^{-1}$. In the SGP, V_Z contains variables $z_{i,k,j}$ for each golfer i, week k, and group j with $D_Z(z_{i,k,j}) = \{0, 1\}$. Figure 2(a) shows the same solution as those in Figure 1, but expressed in V_Z.

4.2 From Value Symmetries to Variable Symmetries

In the rest of the section, we suppose $M_s = (V_s, C_s)$, $M_t = (V_t, C_t)$, and $M_Z = (V_Z, C_Z)$ are CSP models for the same MAP with n aspects, where $V_s = (X_s, D_{X_s})$ and $V_t = (X_t, D_{X_t})$ are aspect viewpoints, and $V_Z = (Z, D_Z)$ is the 0/1 viewpoint.

THEOREM 1. *Given a value symmetry τ under $U_s \subseteq X_s$. If (1) there exists $Obj'(k) \subseteq Obj(k)$ for $1 \leq k \leq n$ and $k \neq s$ such that $U_s = \{x_s[i_1] \cdots [i_{s-1}][i_{s+1}] \cdots [i_n] \mid \bigwedge_{1 \leq k \leq n, k \neq s} i_k \in Obj'(k)\}$, and (2) $Obj'(t) = Obj(t)$, then there is a mapping σ with $\sigma(f_{s,t}(\theta)) = f_{s,t}(\tau(U_s, \theta))$ for all $\theta \in sol(M_s)$, where*

$$\sigma(x_t[i_1] \cdots [i_{t-1}][i_{t+1}] \cdots [i_n]) = \begin{cases} x_t[i_1] \cdots [i_{s-1}][\tau(i_s)][i_{s+1}] \cdots [i_n] & \textit{if } \bigwedge_{\substack{1 \leq k \leq n, \\ k \neq s, k \neq t}} i_k \in Obj'(k) \\ x_t[i_1] \cdots [i_{t-1}][i_{t+1}] \cdots [i_n] & \textit{otherwise.} \end{cases}$$

In addition, σ is a variable symmetry in M_t corresponding to τ in M_s.

Theorem 1 shows that given a value symmetry τ under U_s in V_s, we can find a solution-preserving bijective mapping σ for variables in M_t (i.e., a variable symmetry in V_t) under two conditions. First, the variable subset U_s cannot be arbitrarily chosen. The set of variable indices in U_s has to be the Cartesian product of a subset of the objects in each aspect. Second, $Obj'(t)$ must contain all the objects in aspect t, which corresponds to the domains in V_t.

We illustrate Theorem 1 using the $(3, 2, 3)$ instance of the SGP. Let the golfers, weeks, and groups be the first, second, and third aspect respectively. Hence, $Obj(1) = \{1, \ldots, 6\}$ and $Obj(2) = Obj(3) = \{1, 2, 3\}$. In V_G, any value symmetry is under all the golfers in one week. For example, the value symmetry $\tau(1) = 2$, $\tau(2) = 3$, and $\tau(3) = 1$ is under $U = \{g_{1,1}, \ldots, g_{6,1}\} = \{g_{i,k} \mid i \in Obj'(1) = Obj(1) \land k \in Obj'(2) = \{1\}\}$, the set of all golfers in week 1, which satisfies condition (1) in Theorem 1. Condition (2) is also satisfied because $Obj'(1) = Obj(1)$. Therefore, τ corresponds to a variable symmetry σ in V_P, which uses aspect 1 (golfers) to form the domains, with $\sigma(p_{1,1}) = p_{2,1}$, $\sigma(p_{2,1}) = p_{3,1}$, and $\sigma(p_{3,1}) = p_{1,1}$. On the other hand, $Obj'(2) = \{1\} \neq Obj(2)$. Hence, τ does not correspond to any variable symmetry in V_W, which uses aspect 2 (weeks) to form the domains. Figure 2(b) shows the solution in V_W after applying τ to the solution in Figure 1(a). No variable symmetries can transform the solution in Figure 1(c) to the one in Figure 2(b).

The previous theorem specifies the conditions when a value symmetry in M_s corresponds to a variable symmetry in M_t. The following theorem shows that a value symmetry τ in M_s *always* correspond to a variable symmetry σ in M_Z.

THEOREM 2. *Given a value symmetry τ under $U_s \subseteq X_s$, $\sigma(f_{s,Z}(\theta)) = f_{s,Z}(\tau(U_s, \theta))$ for all $\theta \in sol(M_s)$, where*

$$\sigma(z[i_1]\cdots[i_n]) = \begin{cases} z[i_1]\cdots[i_{s-1}][\tau(i_s)][i_{s+1}]\cdots[i_n] \\ \qquad \text{if } x_s[i_1]\cdots[i_{s-1}][i_{s+1}]\cdots[i_n] \in U_s \\ z[i_1]\cdots[i_n] \\ \qquad \text{otherwise.} \end{cases}$$

In addition, σ is a variable symmetry in M_Z corresponding to τ in M_s.

The value symmetry τ under $U = \{g_{1,1}, \ldots, g_{6,1}\}$ with $\tau(1) = 2$, $\tau(2) = 3$, and $\tau(3) = 1$ corresponds to the variable symmetry σ in V_Z where σ is the identity except $\sigma(z_{i,1,1}) = z_{i,1,2}$, $\sigma(z_{i,1,2}) = z_{i,1,3}$, and $\sigma(z_{i,1,3}) = z_{i,1,1}$ for $1 \leq i \leq 6$.

4.3 Symmetry Breaking Constraints in Two Viewpoints

Recall that variable symmetry breaking constraints are easier to express than value symmetry breaking constraints. By Theorems 1 and 2, value symmetries in a CSP (V, C) can correspond to variable symmetries in another CSP (V', C'). We can thus break the value symmetries in (V, C) by combining (V, C) and $(V', C' \cup C_s)$ using channeling constraints [1], where C_s is the set of variable symmetry breaking constraints in V' for breaking the value symmetries in V. Since (V, C) and (V', C') are models for the same MAP, C' is logically redundant with respect to C and the channeling constraints. Hence, we can drop any of the constraints in C' when we connect V and V'. However, combining mutually redundant models with channeling constraints increases constraint propagation [1]. Therefore, a possible way is to drop only constraints in C' which are propagation redundant [2] so that there would not be less propagation. Note that if we drop *all* the constraints in C', then only (V, C) and (V', C_s) are combined, and V' is solely used for expressing the variable symmetry breaking constraints for the value symmetries in V. Variable symmetries in (V, C), if exist,

$$\overset{M_1}{\begin{pmatrix} 1\,3\,2 \\ 2\,1\,3 \\ 3\,2\,1 \end{pmatrix}} \quad \overset{M_2}{\begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \\ 2\,3\,1 \end{pmatrix}} \quad \overset{M_3}{\begin{pmatrix} 2\,3\,1 \\ 1\,2\,3 \\ 3\,1\,2 \end{pmatrix}} \quad \overset{M_4}{\begin{pmatrix} 2\,1\,3 \\ 3\,2\,1 \\ 1\,3\,2 \end{pmatrix}} \quad \overset{M_5}{\begin{pmatrix} 3\,2\,1 \\ 1\,3\,2 \\ 2\,1\,3 \end{pmatrix}} \quad \overset{M_6}{\begin{pmatrix} 3\,1\,2 \\ 2\,3\,1 \\ 1\,2\,3 \end{pmatrix}}$$

(a)

$$\overset{M'_1}{\begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \\ 3\,1\,2 \end{pmatrix}} \quad \overset{M'_2}{\begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \\ 2\,3\,1 \end{pmatrix}} \quad \overset{M'_3}{\begin{pmatrix} 2\,3\,1 \\ 1\,2\,3 \\ 3\,1\,2 \end{pmatrix}} \quad \overset{M'_4}{\begin{pmatrix} 3\,1\,2 \\ 1\,2\,3 \\ 2\,3\,1 \end{pmatrix}} \quad \overset{M'_5}{\begin{pmatrix} 2\,3\,1 \\ 3\,1\,2 \\ 1\,2\,3 \end{pmatrix}} \quad \overset{M'_6}{\begin{pmatrix} 3\,1\,2 \\ 2\,3\,1 \\ 1\,2\,3 \end{pmatrix}}$$

(b)

Figure 3: All the Six Solutions of Order 3 QEP*, Expressed in V_N and V_R Respectively

can be tackled by variable symmetry breaking constraints in V as well. Now that both variable and value symmetries can now be tackled by symmetry breaking constraints and channeling constraints, we enjoy the best of both worlds.

An important issue of such symmetry breaking technique is the consistency of the symmetry breaking constraints in the two viewpoints V and V'. Two sets of constraints are *consistent* [5] if and only if at least one element in each symmetry class of assignments, defined by the compositions of the symmetries under consideration, satisfies both sets of constraints. In the following, we first give an example of inconsistent symmetry breaking constraints in two viewpoints, and then give theoretical results on how to avoid this inconsistency problem.

The quasigroup existence problem (QEP), "prob003" in CSPLib, is to find an $\mathcal{N} \times \mathcal{N}$ matrix consisting of numbers 1 to $\mathcal{N}$ with no rows and no columns containing the same number more than once. We consider the variant of the problem (QEP*) which further restricts the main ("southeast") diagonal of the matrix to contain the same number. Both the QEP and QEP* are MAPs with three aspects, namely the rows, columns, and numbers. Aspect viewpoint $V_N = (N, D_N)$ uses the rows and columns to form the variables $n_{i,j} \in N$ and the numbers to form the domains $D_N(n_{i,j}) = \{1, \ldots, \mathcal{N}\}$.

The QEP* contains four forms of symmetries. They are (1) the 180° rotation, (2) reflection along the main diagonal, (3) reflection along the main skew ("northeast") diagonal, and (4) the permutation of the numbers in the matrix. Symmetry (1) implies a variable symmetry σ in V_N, with $\sigma(n_{i,j}) = n_{n+1-i,n+1-j}$ for $1 \leq i, j \leq \mathcal{N}$. Symmetry (4) implies $\mathcal{N}! - 1$ value symmetries τ under N in V_N, where $\langle\tau(1), \ldots, \tau(\mathcal{N})\rangle$ is a permutation of $\langle 1, \ldots, \mathcal{N}\rangle$.

Given a sequence $\langle h_1, \ldots, h_{|N|}\rangle$ of N, symmetry (1) can be broken by symmetry breaking constraint $\langle h_1, \ldots, h_{|N|}\rangle \leq_{lex} \langle\sigma(h_1), \ldots, \sigma(h_{|N|})\rangle$. Although there are $\mathcal{N}!$ possible sequences of N, two common ways of generating sequences of a matrix are the row-by-row and column-by-column traversals, giving $\vec{h}_r = \langle n_{1,1}, n_{1,2}, n_{1,3}, \ldots, n_{3,1}, n_{3,2}, n_{3,3}\rangle$ and $\vec{h}_c = \langle n_{1,1}, n_{2,1}, n_{3,1}, \ldots, n_{1,3}, n_{2,3}, n_{3,3}\rangle$ respectively for order 3 QEP* (i.e., $\mathcal{N} = 3$). The corresponding symmetry breaking constraints for τ, after simplifications, are $n_{1,2} < n_{3,2}$ and $n_{2,1} < n_{2,3}$ respectively, which accept different solutions. Figure 3(a) shows all the six solutions of order 3 QEP*. Solutions M_2, M_4, and M_6 satisfy the former constraint, while M_1, M_3, and M_5 satisfy the latter.

By Theorem 1, the value symmetries in V_N become variable symmetries in $V_R = (R, D_R)$, the aspect viewpoint using the numbers and columns to form the variables $r_{k,j} \in R$ and rows to form the domains $D_R(r_{k,j}) = \{1, \ldots, \mathcal{N}\}$. Both the row-by-row and column-by-column traversals of the matrix of variables in R generates, after simplifications, the same symmetry breaking constraints $\langle r_{k,1}, \ldots, r_{k,\mathcal{N}}\rangle \leq_{lex} \langle r_{k+1,1}, \ldots, r_{k+1,\mathcal{N}}\rangle$, or equivalently $r_{k,1} < r_{k+1,1}$, for $1 \leq k < \mathcal{N}$. Figure 3(b) shows the same six solutions as in Figure 3(a), but expressed in V_R. Only M_1' satisfies $r_{k,1} < r_{k+1,1}$, but M_1 violates the variable symmetry breaking constraint $n_{1,2} < n_{3,2}$. Therefore there are no solutions satisfying $r_{k,1} < r_{k+1,1}$ and $n_{1,2} < n_{3,2}$ simultaneously, and hence they are inconsistent symmetry breaking constraints. On the other hand, M_1 satisfies both $r_{k,1} < r_{k+1,1}$ and $n_{2,1} < n_{2,3}$ simultaneously. As we shall see, the last two symmetry breaking constraints are consistent.

We first define several notions which are useful to address the consistency issue for symmetry breaking constraints in two viewpoints. In a symmetry breaking constraint $\vec{h} \leq_{lex} \langle\sigma(h_1), \ldots, \sigma(h_{|X_s|})\rangle$ for a variable symmetry σ in an aspect viewpoint V_s, $\vec{h}$ is an arbitrary linearization of the matrix to a single dimensional sequence. There are $|X_s|!$ possible variable sequences for X_s, and different sequences may generate different variable symmetry breaking constraints in V_s. In the following, we restrict our attention to only the variable sequences generated by aspect priorities. An *aspect priority* in V_s (*resp.* V_Z) is a sequence of aspects which is a permutation of $\{1, \ldots, n\} \setminus \{s\}$ (*resp.* $\{1, \ldots, n\}$). It defines a scanning sequence of the variables X_s in V_s (*resp.* Z in V_Z). A *scanning sequence* of an aspect priority $\langle k_1, \ldots, k_{n-1}\rangle$ of V_s, denoted as $sseq(\langle k_1, \ldots, k_{n-1}\rangle)$, is a sequence $\langle h_1, \ldots, h_{|X_s|}\rangle$ of X_s such that $h_j \equiv x_s[i_1]\cdots[i_{s-1}][i_{s+1}]\cdots[i_n]$, where $j = 1 + \sum_{1\leq l<n}((i_{k_l} - 1) \times \prod_{l<m<n} |Obj(k_m)|)$. Similarly, a scanning sequence $sseq(\langle k_1, \ldots, k_n\rangle)$ of an aspect priority $\langle k_1, \ldots, k_n\rangle$ of V_Z is a sequence $\langle h_1, \ldots, h_{|Z|}\rangle$ of Z such that $h_j \equiv z[i_1]\cdots[i_n]$, where $j = 1 + \sum_{1\leq l\leq n}((i_{k_l} - 1) \times \prod_{l<m\leq n} |Obj(k_m)|)$. A scanning sequence in V_s (*resp.* V_Z) is an aspect-by-aspect traversal of the matrix of variables in V_s (*resp.* V_Z). There are $(n-1)!$ (*resp.* $n!$) possible aspect priorities in V_s (*resp.* V_Z), and hence the same number of possible scanning sequences for the variables in V_s (*resp.* V_Z).

The three aspects in the QEP* give $j = (i_{k_1} - 1) \times |Obj(k_2)| + i_{k_2}$. Let aspects 1, 2, and 3 be the rows, columns, and numbers respectively. In order 3 QEP*, $|Obj(1)| = |Obj(2)| = |Obj(3)| = 3$. The two aspect priorities $\langle 1, 2\rangle$ and $\langle 2, 1\rangle$ in V_N generates the scanning sequences $\vec{h}_r = \langle n_{1,1}, n_{1,2}, n_{1,3}, \ldots, n_{3,1}, n_{3,2}, n_{3,3}\rangle$ and $\vec{h}_c = \langle n_{1,1}, n_{2,1}, n_{3,1}, \ldots, n_{1,3}, n_{2,3}, n_{3,3}\rangle$ respectively, which are the row-by-row and column-by-column traversals of the matrix in V_N. *Selection* of a sequence $\vec{h}$ under a variable set U, $select(\vec{h}, U)$, is a subsequence of $\vec{h}$ retaining only the variables in U. For example, $select(\vec{h}_r, \{n_{1,1}, n_{2,1}, n_{3,1}\}) = \langle n_{1,1}, n_{2,1}, n_{3,1}\rangle$ and $select(\vec{h}_r, \{n_{1,2}, n_{2,2}, n_{3,2}\}) = \langle n_{1,2}, n_{2,2}, n_{3,2}\rangle$.

We are now ready to give a theorem to specify the condition when variable symmetries in V_t, corresponding to value symmetries in V_s, can be broken consistently with the variable symmetries in V_s. Note that the theorem applies to symmetries of indistinguishable values in V_s.

THEOREM 3. *Let σ be a variable symmetry in V_s, σ' be a variable symmetry in V_t corresponding to the symmetry of two indistinguishable values a and b ($a < b$) under U_s in V_s, $\vec{k} = \langle k_1, \ldots, k_{n-2}\rangle$ be any permutation of $\{1, \ldots, n\} \setminus \{s, t\}$, and $\vec{q}$ be any aspect priority in V_t formed by inserting s into $\vec{k}$. If $\langle h_1, \ldots, h_{|X_s|}\rangle = sseq(\langle k_1, \ldots, k_{n-2}, t\rangle)$, then symmetry breaking constraints $\langle h_1, \ldots, h_{|X_s|}\rangle \leq_{lex} \langle\sigma(h_1), \ldots, \sigma(h_{|X_s|})\rangle$ for σ and $\vec{h}_a' \leq_{lex} \vec{h}_b'$ for σ' are consistent, where $\vec{h}_j = select(sseq(\vec{q}), U_j')$ for $j \in \{a, b\}$ and $U_j' = \{x_t[i_1]\cdots[i_{t-1}][i_{t+1}]\cdots[i_n] | i_s = j \wedge x_s[i_1]\cdots[i_{s-1}][i_{s+1}]\cdots[i_n] \in U_s\}$.*

Suppose a symmetry of two indistinguishable values in V corresponds to a variable symmetry in V'. The above theorem states that if we lexicographically order the variables in V' corresponding to the indistinguishable values, then the aspect corresponding to the domain values in V' (aspect t in the theorem) must be least prioritized in V when generating the variable symmetry breaking constraints in V to maintain consistency between the symmetry breaking constraints in V and V'.

For the previous QEP* example, the symmetry breaking constraint, say, $\langle r_{1,1}, \ldots, r_{1,\mathcal{N}}\rangle \leq_{lex} \langle r_{2,1}, \ldots, r_{2,\mathcal{N}}\rangle$, in V_R corresponds to the constraint $\vec{h}_a \leq_{lex} \vec{h}_b$ in the theorem. There are two possible aspect priorities $\langle 2, 3\rangle$ and $\langle 3, 2\rangle$ in V_R. The variable sequence $\langle r_{1,1}, \ldots, r_{1,\mathcal{N}}\rangle$ is the selection of the scanning sequence of both aspect priorities with index value 1 in aspect 3 (the numbers), i.e., $\langle r_{1,1}, \ldots, r_{1,\mathcal{N}}\rangle = select(sseq(\langle 2, 3\rangle), U') = select(sseq(\langle 3, 2\rangle), U')$ where $U' = \{r_{1,1}, \ldots, r_{1,\mathcal{N}}\}$. Similarly for $\langle r_{2,1}, \ldots, r_{2,\mathcal{N}}\rangle$. Therefore, according to Theorem 3, the variable symmetry breaking constraints in V_N must be generated using the scanning sequence of the aspect priority $\langle 2, 1\rangle$, i.e., aspect 1 (the rows) must be least prioritized, to maintain consistency between the symmetry breaking constraints in V_N and V_R. The variable symmetry breaking constraint $n_{2,1} < n_{2,3}$ is generated using the scanning sequence of the aspect priority $\langle 2, 1\rangle$. Therefore it is consistent with the symmetry breaking constraints in V_R.

Consider again the value symmetries in V_G of the SGP. By Theorem 1, they correspond to variable symmetries in V_P. Theorem 3 ensures that the symmetry breaking constraints $\min p_{j,k} < \min p_{j+1,k}$ for $1 \leq j < \mathcal{G}$ and $1 \leq k \leq \mathcal{W}$ in V_P breaks the value symmetries in V_G, and are consistent with the row and column lexicographic ordering constraints in V_G, which are the simplification results of those generated by both aspect priorities $\langle golfer, week\rangle$ and $\langle week, golfer\rangle$ in V_G. The solution in Figure 1 satisfies both types of symmetry breaking constraints.

The consistency issue between an aspect viewpoint V_s and the 0/1 viewpoint V_Z is less complicated. Unlike Theorem 3, which only applies to symmetries of indistinguishable values in V_s, the following theorem applies to any value symmetries.

THEOREM 4. *Let σ be a variable symmetry in V_s, σ' be a variable symmetry in V_Z corresponding to a value symmetry in V_s, and $\vec{k} = \langle k_1, \ldots, k_{n-1}\rangle$ be an aspect priority in V_s. Symmetry breaking constraints $\langle h_1, \ldots, h_{|X_s|}\rangle \leq_{lex} \langle\sigma(h_1), \ldots, \sigma(h_{|X_s|})\rangle$ for σ and $\langle\sigma'(h_1'), \ldots, \sigma'(h_{|Z|}')\rangle \leq_{lex} \langle h_1', \ldots, h_{|Z|}'\rangle$ for σ' are consistent if (1) $\langle h_1, \ldots, h_{|X_s|}\rangle = sseq(\vec{k})$ and (2) $\langle h_1', \ldots, h_{|Z|}'\rangle = sseq(\langle k_1, \ldots, k_{n-1}, s\rangle)$.*

To maintain consistency between the variable symmetry breaking constraints for σ in V_s and σ' in V_Z, the scanning sequence $sseq(\langle k_1, \ldots, k_{n-1}, s\rangle)$ in V_Z is used, i.e., the aspect priority $\langle k_1, \ldots, k_{n-1}\rangle$ in V_s is retained in addition that aspect s is least prioritized in V_Z. Furthermore, the lexicographic order in V_Z is reverse of that in V_s. This is because a smaller-than order in V_s corresponds to a greater-than order in V_Z, and vice versa.

In the SGP, Theorem 4 ensures that the variable symmetry breaking constraints $\langle z_{1,k,j+1}, \ldots, z_{\mathcal{N},k,j+1}\rangle \leq_{lex} \langle z_{1,k,j}, \ldots, z_{\mathcal{N},k,j}\rangle$ for $1 \leq j < \mathcal{G}$ and $1 \leq k \leq \mathcal{W}$ in V_Z break the value symmetries in V_G, and are consistent with those variable symmetry breaking constraints in V_G.

5. EXPERIMENTS

We test our implementations on the SGP and QEP* to demonstrate the feasibility of our proposal. The experiments, run using ILOG Solver 4.4 [8] on a Sun Blade 1000 workstation with 2GB memory, aim to compare breaking value symmetries using multiple viewpoints and channeling constraints against using the if-then constraints for symmetries of indistinguishable values. We report the number of fails and CPU time (in seconds), with the best of each among the models for each instance highlighted in bold.

We build an integer model of the SGP in V_G, in which the row and column lexicographic ordering constraints in V_G (for symmetries (1) and (2)) are expressed. Using this basis, the int-bool and int-set models use multiple viewpoints and break the value symmetries in V_G (symmetry (3)) as variable symmetries in V_Z and V_G respectively. We perform extensive experiments using various instances and present only those with significant runtimes. Table 1 shows the experimental results of solving for all solutions using different models. A cell labeled with "-" means that the search does not terminate in 2 hours of CPU time. The int-bool model achieves less propagation than the if-then and int-set models do. However, its performance is much better than the if-then model in most instances. The int-set model has the same number of fails as the if-then model, but is generally much faster due to the inefficient execution of the if-then constraints. The int-set and int-bool models are incomparable. The former is sometimes slightly slower than the latter, but in certain instances (e.g., (5, 5, 3), (5, 5, 4), (5, 5, 5), (6, 6, 3), and *etc.*), the difference in number of fails between them is so large that the int-set model shows its robustness and is significantly faster.

Another approach to break value symmetries is to develop global constraints for them. In particular, we develop global constraints to maintain *value precedence* [9] which breaks symmetries of indistinguishable values. We perform experiments on models using the value precedence global constraints as well. It is not surprising that such models perform better than the int-bool and int-set models, because specialized propagation algorithms are used to implement the global constraints. The advantage of using multiple viewpoints, however, is simplicity of and readiness for use in existing constraint programming systems.

We also perform experiments on a set model of the SGP in V_P with the value symmetries broken as variable symmetries in V_G, as well as on an integer model of the QEP*. We obtain similar results as those in Table 1, but due to space limitations, we skip the details.

6. CONCLUDING REMARKS

We show how value symmetries can be tackled effectively and efficiently as variable symmetries with the help of multiple viewpoints and channeling constraints. An advantage of our approach is that it is readily deployable in existing constraint programming systems, without having to invent and implement a specialized propagation algorithm, such as the value precedence constraint [9].

Table 1: Experimental Results for the Social Golfer Problem, using Integer Variables

	int-bool		int-set		if-then	
g, s, w	fails	time	fails	time	fails	time
5, 2, 4	52543	**57.9**	**36804**	74.3	**36804**	74.03
5, 2, 5	867791	**1075.95**	**758610**	1458.34	**758610**	1400.16
5, 2, 6	6605552	**6839.61**	-	-	-	-
5, 2, 9	9166800	**3210.56**	**8325932**	4073.58	**8325932**	4326.68
5, 3, 3	213328	**192.31**	**207217**	269.96	**207217**	368.98
5, 3, 7	10019241	**3821.9**	**10954130**	6320.45	-	-
5, 4, 3	382664	183.63	**126170**	**120.58**	**126170**	183.79
5, 5, 3	21038	13.32	**42**	**1.22**	**42**	1.94
5, 5, 4	190084	93.7	**9031**	**8.6**	**9031**	15.91
5, 5, 5	27746	14.26	**1933**	**2.58**	**1933**	5.01
5, 5, 6	1776	1.26	**237**	**0.45**	**237**	0.88
6, 2, 3	110529	**95.63**	**39059**	119.85	**39059**	140.87
6, 6, 3	-	-	**20917**	**1528.85**	**20917**	3300.59
7, 3, 2	7504	**63.26**	**180**	91.5	**180**	189.85
7, 4, 2	66985	**332.42**	**60747**	506.17	**60747**	1234.79
7, 5, 2	131666	145.86	**46007**	**123.71**	**46007**	365.82
7, 6, 2	29485	**31.18**	**16447**	36.46	**16447**	128.48

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