Tutorial 11: Expectation and Variance of linear combination of random variables

Fact 1:
For random variable X:

a) $E[aX+b] = aE[X]+b$

b) $Var[aX+b] = a^2\,Var[X]$

Fact 2:
For random variables $X_1, X_2, \ldots, X_n$:

a) The following equation holds for arbitrary random variables $X_1, X_2, \ldots, X_n$

$$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

b) If $X_1, X_2, \ldots, X_n$ are independent, then

$$Var[X_1 + X_2 + \ldots + X_n] = Var[X_1] + Var[X_2] + \ldots + Var[X_n]$$

Fact1 + Fact2 $\Rightarrow$ Fact 3:
For random variables $X_1, X_2, \ldots, X_n$ and arbitrary constant $c_0, c_1, \ldots, c_n$:

a) The following equation holds for arbitrary random variables $X_1, X_2, \ldots, X_n$

$$E[c_0 + c_1X_1 + c_2X_2 + \ldots + c_nX_n] = c_0 + c_1E[X_1] + c_2E[X_2] + \ldots + c_nE[X_n]$$

b) If $X_1, X_2, \ldots, X_n$ are independent, then

$$Var[c_0 + c_1X_1 + c_2X_2 + \ldots + c_nX_n] = c_1^2\,Var[X_1] + c_2^2\,Var[X_2] + \ldots + c_n^2\,Var[X_n]$$

Notes: The facts hold for both continuous and discrete random variables.

Proof of Fact 1:

a) Let $g(X) = aX + b$

$$\begin{aligned} E[g(X)] &= \int (ax+b) f_X(x) dx \\ &= a\int x f_X(x) dx + b\int f_X(x) dx \\ &= aE[X] + b \end{aligned}$$

b) Let $g(X) = aX + b$

$$\begin{aligned} Var[g(X)] &= E[g(X)^2] - E[g(X)]^2 \\ &= E[(aX+b)^2] - (aE[X]+b)^2 \\ &= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2 \\ &= a^2E[X^2] + 2abE[X] + b^2 - (aE[X]+b)^2 \\ &= a^2(E[X^2] - E[X]^2) \\ &= a^2 Var[X] \end{aligned}$$

Proof of Fact 2:

a) Prove by induction.

- First prove for arbitrary two random variable X, Y (note we don't make independence assumption here), $E[X+Y] = E[X] + E[Y]$:
  Denote $f(x,y)$ the joint probability density function of X, Y.

$$\begin{aligned} E[X+Y] &= \int_y \int_x (x+y) f(x,y) dx dy \\ &= \int_x x \int_y f(x,y) dy dx + \int_y y \int_x f(x,y) dx dy \\ &= \int_x x f_X(x) dx + \int_y y f_Y(y) dy \\ &= E[X] + E[Y] \end{aligned}$$

- Suppose

$$E[\sum_{i=1}^{k-1} X_i] = \sum_{i=1}^{k-1} E[X_i]$$

  Define random variable $Y_{k-1} = \sum_{i=1}^{k-1} X_i$, then

$$\begin{aligned} E[\sum_{i=1}^{k} X_i] &= E[Y_{k-1} + X_k] \\ &= E[Y_{k-1}] + E[X_k] \\ &= \sum_{i=1}^{k-1} E[X_i] + E[X_k] \\ &= \sum_{i=1}^{k} E[X_i] \end{aligned}$$

b) Prove by induction

Problems:

a) $X_i, i = 1, \dots, n$ are independent normal variables with respective parameters μ_i and σ_i^2, then $X = \sum_{i=1}^{n} X_i$ is normal distribution, show that expectation of X is $\sum_{i=1}^{n} \mu_i$ and variance is $\sum_{i=1}^{n} \sigma_i^2$.

b) A random variable X with gamma distribution with parameters $(n, \lambda), n \in N, \lambda > 0$ can be expressed as sum of n independent exponential random variables: $X = \sum_{i=1}^{n} X_i$, here X_i are independent exponential random variable with the same parameter λ. Calculate expectation and variation of gamma random variable X.

c) A random variable X is named χ_n^2 distribution with if it can be expressed as the squared sum of n independent standard normal random variable: $X = \sum_{i=1}^{n} X_i^2$, here X_i are independent standard normal random variable. Calculate expectation of random variable X.

d) $X_i, i = 1, \dots n$ are independent uniform variables over interval $(0, 1)$. Calculate the expectation and variation of the random variable $X = \frac{1}{n} \sum_{i=1}^{n} X_i$.

Solution:

a) From Fact 2

b) $E[X] = \sum_{i=1}^{n} E[X_i] = n/\lambda$
$Var[X] = \sum_{i=1}^{n} Var[X_i] = n/\lambda^2$

c) $E[X_i^2] = Var[X_i] = 1$ (Recall $Var[X] = E[X^2] - E[X]^2$)
$E[X] = \sum_{i=1}^{n} E[X_i^2] = n$

d) $E[X] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{1}{n} \frac{n}{2} = \frac{1}{2}$
$Var[X] = \frac{1}{n^2} \sum_{i=1}^{n} Var[X_i] = \frac{1}{n^2} \cdot n \cdot \frac{1}{12} = \frac{1}{12n}$
We can see the variance diminishes as $n \to \infty$.