Homework # 5 Solution

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1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) What is the value of c?
(b) What is the cumulative distribution function of X?

Solution:

(a) $1 = \int_{-\infty}^{+\infty} f(x)dx = \int_{-1}^{1} c(1-x^2)dx = c(x - \frac{1}{3}x^3)\,\big|_{-1}^{1} = \frac{4}{3}c \Rightarrow c = \frac{3}{4}$

(b) $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-1}^{x} \frac{3}{4}(1-t^2)dt = \frac{3}{4}(x - \frac{1}{3}x^3)\,\big|_{-1}^{x} = \frac{3}{4}(x - \frac{x^3}{3} + \frac{2}{3})$

2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x > 0 \\ 0 & x \leq 0 \end{cases} \quad (2)$$

What is the probability that the system functions for at least 5 months?

Solution:

$1 = \int_{0}^{+\infty} Cxe^{-x/2} = -C(2x+4)e^{-x/2}\,\big|_{0}^{+\infty} = 4C \Rightarrow C = 1/4$

$P(X > 5) = \int_{5}^{+\infty} \frac{1}{4}xe^{-x/2}dx = -\frac{1}{4}(2x+4)e^{-x/2}\,\big|_{5}^{+\infty} = \frac{7}{2}e^{-5/2}$

3. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

What must the capacity of the tank be so that the probability of the supply's being exhausted in a given week is .01?

Solution:

Let c be the capacity of the tank, then c must satisfy

$0.01 = P(X \leq c) = \int_{c}^{1} 5(1-x)^4 dx = (1-c)^5$

$\Rightarrow c = 1 - \sqrt[5]{0.01} \approx 0.6$

4. A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over (0,100). There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25,50, and 75 miles, respectively, from A, Do you agree? Why?

Solution:

One way to compare the two strategies: Compare the expected distance to the nearest bus service station when the bus break down.

Denote Y the distance to the nearest bus service station when the bus break down and S the No. of the nearest bus service station (No. 1, 2, 3):

a) In the original setting that the three bus service stations numbered 1, 2, 3 are 0, 50 and 100 miles away from A:

$$\begin{aligned} E[Y] &= E[Y|S=1]P(S=1) + E[Y|S=2]P(S=2) + E[Y|S=3]P(S=3) \\ &= (\int_0^{25} |x-0|dx) \cdot \frac{25-0}{100} + (\int_{25}^{75} |x-50|dx) \cdot \frac{75-25}{100} + (\int_{75}^{100} |x-100|dx) \cdot \frac{100-75}{100} \\ &= 12.5 \end{aligned}$$

b) In the suggested setting that the three bus service stations numbered 1, 2, 3 are 25, 50 and 100 miles away from A:

$$\begin{aligned} E[Y] &= E[Y|S=1]P(S=1) + E[Y|S=2]P(S=2) + E[Y|S=3]P(S=3) \\ &= (\int_0^{37.5} |x-25|dx) \cdot \frac{37.5-0}{100} + (\int_{37.5}^{62.5} |x-50|dx) \cdot \frac{62.5-37.5}{100} + (\int_{62.5}^{100} |x-75|dx) \cdot \frac{100-75}{100} \\ &\approx 10 \end{aligned}$$

Therefore, the suggested positions are better.

5. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P\{X > c\} = .10$.

Solution:
Standard trick: Transfer the normal random viable X to standard normal random viable $(X-\mu)/\sigma$

$$\begin{aligned} P(X > c) &= P((X-12)/2 > (c-12)/2) \\ &= 1 - \Phi((c-12)/2) \\ &= 0.10 \end{aligned}$$

From textbook table 5.1, we know:

$$1 - \Phi(1.28) \approx 0.10$$

Solve equation

$$1.28 = (c-12)/2c$$
$$\Rightarrow c = 14.56$$

6. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year-old man are over 6 feet, 2 inches tall? What percentage of men in the 6-footer club are over 6 feet, 5 inches?

Solution:
Denote X the height of a 25-year-old man.

(a) *1 foot = 12 inches, therefore 6 feet 2 inches = 74 inches*

$$\begin{aligned} P(X > 74) &= P((X-71)/2.5 > 1.2) \\ &= 1 - \Phi(1.2) \\ &= 1 - 0.8849 \quad \text{(textbook table 5.1)} \\ &\approx 11.5\% \end{aligned}$$

(b) *6 feet = 72 inches, 6 feet 5 inches = 77 inches*

$$\begin{aligned} P(X > 77|X > 72) &= P((X-77)/2.5 > 2|(X-72)/2.5 > 0.4) \\ &= (1-\Phi(2))/(1-\Phi(0.4)) \\ &= (1-0.9861)/(1-0.6554) \quad \text{(textbook table 5.1)} \\ &\approx 4\% \end{aligned}$$

7. A model for the movement of a stock supposes that if the present price of the stock is s, then, after one period, it will be either us with probability p or ds with probability $1-p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30 percent after the next 1000 periods if $u = 1.012$, $d = 0.990$, and $p = 0.52$.

Solution:
Let s be the initial price of the stock. Denote X the number of increase periods among the 1000 time periods. Then the price at the end is

$$su^X d^{1000-X}$$

In order for thye price to be at least $1.3s$, we need

$$d^{1000}(\frac{u}{d})^X > 1.3$$

$$\Rightarrow X > \frac{\log(1.3) - 1000 \log d}{\log(u/d)} \approx 469.2$$

That is, we need at least 470 increase periods.
Since X is binomial with parameters 1000, 0.52, we have

$$\begin{aligned}
\text{(continuity correction here)} \quad P(X > 469.5) &= P(\frac{X - 1000 \cdot 0.52}{\sqrt{1000 \cdot 0.52 \cdot 0.48}} > \frac{469.5 - 1000 \cdot 0.52}{\sqrt{1000 \cdot 0.52 \cdot 0.48}}) \\
&\approx \Phi(-3.196) \quad \text{(The \textbf{DeMoivre-Laplace limit theorem}, textbook p.204)} \\
&\approx 0.9993
\end{aligned}$$