Homework # 1 Solutions

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1. We have three coins: two nickels (5 cents coin) and one dime (10 cents coin). Each coin has two sides: one is head and the other side shows the worth of the coin. You pay $0.1 to enter the game to toss these three coins. You will receive those coins which fall heads up.

   (a) What is the sample space $\mathcal{S}$ of this problem?

   (b) What is the probability of each sample point in $\mathcal{S}$?

   (c) What is the probability of winning some money?

   (d) What is the probability of losing some money?

   (e) What is the probability of breaking even?

   Solution: We mark the coins so that we can distinguish between the two nickels. The nickels are given numbers 1 and 2, and the dime is given number 3.

   (a) The sample space is $\mathcal{S} = \{(1H, 2H, 3H), (1H, 2H, 3T), (1H, 2T, 3H), (1H, 2T, 3T), (1T, 2H, 3H), (1T, 2H, 3T), (1T, 2T, 3H), (1T, 2T, 3T)\}$.

   (b) The probability of each sample point in $\mathcal{S}$ is $\frac{1}{8}$, since each of the outcome is equally likely.

   (c) Let E_1 be the event of winning some money. Then, $E_1 = \{(1H, 2H, 3H), (1H, 2T, 3H), (1T, 2H, 3H)\}$. Thus, $P(E_1) = \frac{3}{8}$.

   (d) Let E_2 be the event of losing some money. Then, $E_2 = \{(1T, 2T, 3T), (1H, 2T, 3T), (1T, 2H, 3T)\}$. Thus, $P(E_2) = \frac{3}{8}$.

   (e) Let E_3 be the event of breaking even. Then, $E_3 = \{(1H, 2H, 3T), (1T, 2T, 3H)\}$. Thus, $P(E_3) = \frac{2}{8} = \frac{1}{4}$.

2. We keep tossing a coin until it lands head up.

   (a) What is the sample space $\mathcal{S}$?

   (b) What is the probability of only tossing the coin once?

   (c) What is the probability of only tossing the coin twice?

   Solution:

   (a) The sample space is $\mathcal{S} = \{0, 1, 2, ...\}$, where each element denotes the number of tails before the first head appears.

   (b) The probability of only tossing the coin once is $P(\{0\}) = \frac{1}{2}$.

   (c) The probability of only tossing the coin twice is $P(\{1\}) = \frac{1}{4}$.

3. We want to count the number of car accidents in Hong Kong in a year. What is the sample space $\mathcal{S}$?

   Solution: Note that the number of accidents is not necessarily limited by the number of cars in Hong Kong. A car can be involved in more than one accident. The sample space $\mathcal{S} = \{0, 1, 2, ...\}$, where each element denotes the number of accidents in Hong Kong in a year.

4. I see an opened can of coca-cola, and want to find out the probability that it is less than $1/3$ filled. What is the sample space $\mathcal{S}$ of this problem?

   Solution: Suppose the volume of the can is V. The sample space $\mathcal{S} = \{v | v \in \mathbf{R}, 0 \leq v \leq V\}$, where v is the volume filled at the observation point.
   Another way to view this problem is to define the sample space $\mathcal{S} = \{q | q \in \mathbf{R}, 0 \leq q \leq 1\}$, where q is the fraction of volume which is filled in at the observation point.

5. Let $A = \{1, 2, 3, 4, 5\}$. Consider A_3 is a 3-element subset of A.

   (a) What are all possible elements of A_3?

   (b) If A has n distinct elements, what is the size of the subset A_k, where $k \leq n$?

   Solution:

   (a) The possible elements in A_3 are: $\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}$.

   (b) The size of the subset A_k is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

6. Four people are randomly seated in a row, we want to find the probability that you are sitting beside your girlfriend (or boyfriend).

   (a) What is the sample space $\mathcal{S}$?

   (b) What is the probability you are sitting next to your love one?

   (c) Instead of four people, we have ten people sitting in a row. What is the probability you are sitting next to your love one?

   (d) Instead of ten people sitting in a row, we have ten people sitting in a *circle*.

      - What is the state space $\mathcal{S}$ of this problem?
      - What is the probability you are sitting beside your love one?

   Solution: (Note that there can be different angles to view this problem. The sample space may not be identical. The following is only one possible solution.)

   (a) We denote the positions in the low by number 1 to 4, and denote the four people by A, B, C and D. The sample space is all possible arrangement to seat people into the four positions. The size of the sample space is $4! = 24$.

   (b) Suppose you are A and your love one is B. Denote E_b as the event that you sit next to your love one. To sit next to each other, you and your love one need to choose two neighboring positions first (the number of possible choice is $\binom{3}{1}$), then arrange the positions of you two (the number of possible choices is 2), and lastly arrange the positions for the rest two people (the number of possible choices is 2!). In all, the total number of outcomes in E_b is $\binom{3}{1} \times 2 \times 2! = 12$. Therefore, $P(E_b) = 12/24 = 0.5$.

   (c) Similar to previous, define the sample space $\mathcal{S}$ as all possible arrangements of the ten people. The size of the sample space is 10!. Denote E_c as the event that you are sitting next to your love one. The total number of outcomes in E_c is $\binom{9}{1} \times 2 \times 8!$. Therefore, $P(E_c) = \frac{\binom{9}{1} \times 2 \times 8!}{10!} = \frac{1}{5}$.

   (d) Denote E_d as the event that you sit next to your love one in a circle. We still define the sample space $\mathcal{S}$ as all possible arrangements of the ten people. The size of the sample space is 10!. Different from a row, when we consider all possible arrangements that you sit next to your loved one, we can find 10 pairs of neighboring positions: $(1,2), (2,3), ..., (9,10), (10,1)$. Therefore, the number of outcomes that you sit next to your love one is $\binom{10}{1} \times 2 \times 8!$. $P(E_d) = \frac{\binom{10}{1} \times 2 \times 8!}{10!} = \frac{2}{9}$.

7. Assume in the CSE Department, the probability that a computer is of Window OS or Mac OS is equally likely. You have three computers in front of you.

   - What is the state space $\mathcal{S}$ of this problem?
   - What is the probability you have exact one computer with Mac OS?
   - Assume that the probability that a computer is of Window OS is 0.9 and Mac OS is 0.1. You have three computers in front of you, what is the probability of having exactly two computers with Mac OS?

   Solution: Denote by W a computer with Windows OS, and by M a computer with Mac OS.

   - Define an outcome as the three computers in front of you, from the left hand side to the right hand side. The sample space $\mathcal{S} = \{MMM, MMW, MWM, MWW, WMM, WMW, WWM, WWW\}$.
   - The outcomes in the sample space are equally likely. There are three events in the sample space to have exact one computer with Mac OS. Therefore, the probability you have exact one computer with Mac OS is 3/8.
   - The probability of having exactly two computers with Mac OS is $P = \binom{3}{2} \times (0.1)^2 \times 0.9 = 0.027$.