1. The body temperatures of a healthy person and an infected person are Normal$(36.8, 0.5)$ and Normal$(37.8, 1.0)$ random variables, respectively. About 1% of the population is infected.

   (a) What is the conditional probability that I am infected given that my temperature is t?

   (b) For which values of t am I more likely to be infected than not?

   Solution:

   (a) Let A be the event that I am infected, and T be my body temperature. By the total probability theorem,

   $$f_T(t) = \mathrm{P}(A) f_{T|A}(x) + \mathrm{P}(A^c) f_{T|A^c}(t),$$

   where $T|A$ is a Normal$(37.8, 1.0)$ random variable and $T|A^c$ is a Normal$(36.8, 0.5)$ random variable. The (unconditional) PDF of X is

   $$f_T(t) = \frac{0.01}{\sqrt{2\pi}} e^{-\frac{(t-37.8)^2}{2}} + \frac{0.99}{\sqrt{2\pi}(0.5)} e^{-\frac{(t-36.8)^2}{2(0.5)^2}} = \frac{0.01}{\sqrt{2\pi}} e^{-(t-37.8)^2/2} + \frac{1.98}{\sqrt{2\pi}} e^{-2(t-36.8)^2}.$$

   By Bayes' rule, the conditional probability of A given T is

   $$P(A|T=t) = \frac{P(A) f_{T|A}(t)}{f_T(t)} = \frac{0.01 e^{-(t-37.8)^2/2}}{0.01 e^{-(t-37.8)^2/2} + 1.98 e^{-2(t-36.8)^2}}$$

   (b) I am more likely to be infected than not when $\mathrm{P}(A) > \mathrm{P}(A^c)$, namely when

   $$0.01 e^{-(t-37.8)^2/2} > 1.98 e^{-2(t-36.8)^2}.$$

   Taking logarithms of both sides this is equivalent to a quadratic inequality in t. Solving this inequality, we obtain that $\mathrm{P}(A) > \mathrm{P}(A^c)$ holds when $t < t_-$ or $t > t_+$, where $t_- \approx 34.4742$ and $t_+ \approx 38.4591$.

2. A coin has probability P of being heads, where P itself is a Uniform$(0, 1)$ random variable. The coin is flipped twice. Given that it comes out heads both times, what is the (posterior)

   (a) PDF of P?

   (b) expected value of P?

   (c) probability that the next two flips are both heads?

   Solution:

   (a) Let X be the number of heads. By Bayes' rule,

   $$f_{P|X}(p|x) = \frac{f_{X|P}(x|p) f_P(p)}{f_X(x)},$$

   where $f_{P|X}(\cdot|x)$ is the conditional PDF of P given x heads were observed, $f_{X|P}(x|p) = \binom{2}{x} p^x (1-p)^{2-x}$ is the conditional p.m.f. of X given the coin has bias p, $f_P(p)$ is the (prior) PDF of the Uniform$(0,1)$ random variable P, and f_X is the p.m.f. of X. Plugging in these formulas, we get that

$$f_{P|X}(p|2) = \frac{p^2 \cdot 1}{f_X(x)} = \frac{p^2}{f_X(x)},$$

when $0 \leq p \leq 1$. Since $f_{P|X}(p|2)$ is a PDF it must integrate to one, so

$$f_X(x) = \int_0^1 p^2 dp = \frac{1}{3}$$

and so $f_{P|X}(p|2) = 3p^2$.

(b) The expected value of P given $X = 2$ is

$$\mathrm{E}[P|X=2] = \int_0^1 p \cdot f_{P|X}(p|2) dp = \int_0^1 3p^3 dp = \frac{3}{4}.$$

(c) Let Y be the number of heads in the next two flips, By Bayes' rule,

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\int_0^1 f_{XY|P}(x,y|p) f_P(p) dp}{f_X(x)},$$

where $f_{XY|P}(x,y|p) = \binom{2}{x} p^x (1-p)^{2-x} \cdot \binom{2}{y} p^y (1-p)^{2-y}$ is the conditional joint p.m.f. of X, Y given the coin has bias p, $f_P(p)$ is the (prior) PDF of the Uniform$(0,1)$ random variable P, and $f_X(x) = 1/3$. Plugging in these formulas, we get that

$$f_{Y|X}(2|2) = \frac{\int_0^1 f_{XY|P}(2,2|p) f_P(p) dp}{1/3} = 3 \int_0^1 p^4 dp = \frac{3}{5}.$$

3. Raindrops hit your head at a rate of 1 per second. What is the PDF of the time at which the second raindrop hits you? How about the third one? (**Hint:** convolution)

Solution: The time before the second raindrop is $Y = X_1 + X_2$, where X_1 and X_2 are independent Exponential(1) random variables. We calculate the PDF of Y using the convolution formula:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(y - x_1) dx_1 = \int_0^y e^{-x_1} e^{-y+x_1} dx_1 = y e^{-y}.$$

The third raindrop hits at time $Z = Y + X_3$, where X_3 is another independent Exponential(1) random variable. By the convolution formula again,

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_{X_3}(z-y) dy = \int_0^z y e^{-y} e^{-z+y} dy = \frac{z^2}{2} e^{-z}.$$

4. In this question you will calculate the PDF of a *product* XY of two independent Uniform$(0, 1)$ random variables X and Y.

   (a) What is the PDF of $X' = \ln X$?

   (b) What is the PDF of $Z = \ln X + \ln Y$?

   (c) What is the PDF of $e^Z = XY$?

Solution:

(a) The PDF of X' is

$$f_{X'}(x') = \frac{d}{dx'}\mathrm{P}(X' \le x') = \frac{d}{dx}\mathrm{P}(X \le e^{x'}) = \frac{d}{dx'}e^{x'} = e^{x'}$$

when $x' \le 0$ (that is, $0 < e^{x'} \le 1$) and zero otherwise.

(b) Let $Y' = \ln Y$. By the convolution formula,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X'}(x') f_{Y'}(z - x') dx = \int_z^0 e^{x'} e^{z-x'} dx' = -ze^z$$

when $z \le 0$ and zero otherwise.

(c) The PDF of $Z' = e^Z$ is

$$f_{Z'}(z') = \frac{d}{dz'}\mathrm{P}(Z' \le z') = \frac{d}{dz'}\mathrm{P}(Z \le \ln z') = \frac{1}{z'} f_Z(\ln z') = -\ln z'$$

when $0 < z' \le 1$ (that is, $\ln z' \le 0$) and zero otherwise.