1. A point is chosen uniformly at random inside a circle with radius 1. Let X be the distance from the point to the centre of the circle. What is the (a) CDF (b) PDF (c) expected value and (d) variance of X? *[Adapted from textbook problem 3.2.7]*

   Solution:

   (a) The PDF of the point is uniform over the circle which has area π, so it has value $1/\pi$ inside the center and zero outside. The event $X \leq x$ consists of all the points in the circle that are at distance less than or equal to x from the center, which is itself a circle of radius x. Therefore the CDF is $\mathrm{P}(X \leq x) = 1/\pi \times x^2\pi = x^2$, and the PDF is $f_X(x) = d\,\mathrm{P}(X \leq x)/dx = 2x$ for $0 \leq x \leq 1$.

   (b) The expected value of X is $\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} 2x^2 = \frac{2}{3}x^3 \mid_0^1 = \frac{2}{3}$.

   (c) The variance of X is $\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2$, where

   $$\mathrm{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{-\infty}^{\infty} 2x^3 = \tfrac{1}{2}x^4 \mid_0^1 = \frac{1}{2}.$$

   Therefore, $\mathrm{Var}(X) = 14 - (\frac{2}{3})^2 = \frac{1}{18}$.

2. Bob's arrival time at a meeting with Alice is X hours past noon, where X is a random variable with PDF

   $$f(x) = \begin{cases} cx, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the value of the constant c.

   (b) What is the probability that Bob arrives by 12.30?

   (c) What is the expected hour of Bob's arrival?

   (d) Given that Bob hasn't arrived by 12.30, what is the probability that he arrives by 12.45?

   (e) Given that Bob hasn't arrived by 12.30, what is the expected hour of Bob's arrival?

   Solution:

   (a) By the axioms of probability, $\int_{-\infty}^{\infty} f(x)dx = 1$. Since

   $$\int_{-\infty}^{\infty} f(x) = \tfrac{1}{2}cx^2 \mid_0^1 = \tfrac{1}{2}c,$$

   c must be equal to 2.

   (b) The CDF of X is $F(x) = \mathrm{P}(X \leq x) = \int_{-\infty}^{\infty} f(x)dx = x^2$. In particular, $\mathrm{P}(X \leq 0.5) = 0.25$.

   (c) The expected value is $\mathrm{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} 2x^2dx = \frac{2}{3}x^3 \mid_0^1 = \frac{2}{3}$. So Bob is expected to arrive at 12.40.

   (d) The probability that Bob hasn't arrived by 12:30 is $P(X > 0.5) = 1 - P(X \leq 0.5) = 1 - 0.25 = 0.75$. The probability that Bob hasn't arrived by 12:30 but arrives by 12:45 is $P(0.75 \geq X > 0.5) = P(X \leq 0.75) - P(X \leq 0.5) = F(0.75) - F(0.5) = \frac{9}{16} - \frac{1}{4} = \frac{5}{16}$. Therefore, the conditional probability is

   $$\mathrm{P}(X \leq 0.75 \mid X > 0.5) = \frac{\mathrm{P}(0.75 \geq X > 0.5)}{\mathrm{P}(X > 0.5)} = \frac{5/16}{1 - 1/4} = \frac{5}{12}$$

   .

(e) Given that Bob hasn't arrived by 12:30, the probability that Bob arrives by x hour past noon is

$$P(X \le x \mid X > 0.5) = \frac{P(x \ge X > 0.5)}{P(X > 0.5)} = \begin{cases} \frac{x^2 - 0.25}{0.75} & 0.5 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the conditional PDF is

$$f(x) = \begin{cases} \frac{8}{3}x & 0.5 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and the expected value is $\int_{-\infty}^{\infty} x f(x) = \int_{0.5}^{1} \frac{8}{3}x^2 = \frac{8}{9}x^3|_{0.5}^{1} = \frac{7}{9}$.

3. Alice arrives at her bus stop at noon. Buses arrive at a rate of 3 per hour in the next hour and 1 per hour after that.

(a) Divide each hour into n equal intervals and let E_i be the event "a bus arrives in the i-th interval past noon." What is $P(E_i)$? (Assume n is sufficiently large so that the probability of two or more buses arriving in interval i is negligible.)

(b) Let I_n be the index of the interval in which the first bus arrives. Assuming the events E_i are independent, what is the CDF of I_n?

(c) Let T be a random variable whose CDF is $F_T(t) = \lim_{n\to\infty} P(I_n/n \le t)$. Calculate the CDF of T. What does T represent?

(d) Calculate the PDF and the expected value of T.

Solution:

(a) If $i \le n$, then the event E_i means that the bus will arrive at some interval at the first hour after noon, and $P(E_i) = \frac{3}{n}$. If $i > n$, then the event E_i means that the bus will arrive at some interval after the first hour, and $P(E_i) = \frac{1}{n}$. Therefore,

$$P(E_i) = \begin{cases} \frac{3}{n} & 1 \le i \le n \\ \frac{1}{n} & i > n \end{cases}$$

(b) The event $I_n = t$ happens if E_i does not happen for $i < t$ and E_t happen. Since each event E_i are independent,

$$P(I_n = t) = \begin{cases} (1 - \frac{3}{n})^{t-1}\frac{3}{n} & \text{if } 1 \le t \le n \\ (1 - \frac{3}{n})^{n}(1 - \frac{1}{n})^{t-n-1}\frac{1}{n} & \text{if } t > n \end{cases}$$

The CDF of I_n is

$$P(I_n \le t) = \begin{cases} 1 - (1 - \frac{3}{n})^{t} & \text{if } 1 \le t \le n, \\ (1 - \frac{3}{n})^{n}(1 - (1 - \frac{1}{n})^{t-n}) & \text{if } t > n. \end{cases}$$

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(c) First we compute the probability

$$P(I_n/n \le t) = P(I_n \le nt) = \begin{cases} 1 - (1 - \frac{3}{n})^{nt} & \text{if } \frac{1}{n} \le t \le 1, \\ (1 - \frac{3}{n})^{n}(1 - (1 - \frac{1}{n})^{nt-n}) & \text{if } t > 1. \end{cases}$$

Taking the limit $n \to \infty$, we get $\lim_{n\to\infty}(1 - \frac{a}{n})^n = \frac{1}{e^a}$. Therefore, we have

$$F_T(t) = \begin{cases} 1 - e^{-3t} & \text{if } 0 \le t \le 1, \\ e^{-3}(1 - e^{-(t-1)}) & \text{if } t > 1. \end{cases}$$

The random variable T represents the arrival time of the first bus.

(d) The PDF of T is

$$f_T(t) = \begin{cases} 3e^{-3t} & \text{if } 0 \le t \le 1, \\ e^{-2}e^{-t} & \text{if } t > 1. \end{cases}$$

Therefore, the expected value of T is

$$\begin{aligned} \mathrm{E}[T] &= \int_0^1 3te^{-3t}dt + \int_1^\infty e^{-2}te^{-t} \\ &= (-(t+\frac{1}{3})e^{-3t})|_0^1 + (e^{-2}(-t-1)e^{-t})|_1^\infty \\ &= \frac{1}{3} + \frac{2}{3}e^{-3}. \end{aligned}$$

4. To send a message $m \in \{-1, 1\}$ to Bob, Alice emits a signal mx of "strength" $x > 0$. Owing to noise Bob receives a Normal$(mx, 1)$ random variable Y and decodes it to the sign of Y ($+1$ if Y is positive, -1 if negative). The cost of operating this scheme is x cents if the decoding is correct and $x + 10$ cents if it isn't. How should Alice pick x to minimize the expected cost?

Solution: Let X be the cost of operating this scheme. By the conditional expectation formula,

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[X \,|\, \operatorname{sign} Y \ne m]\,\mathrm{P}(\operatorname{sign} Y \ne m) + \mathrm{E}[X \,|\, \operatorname{sign} Y = m]\,\mathrm{P}(\operatorname{sign} Y = m) \\ &= (x+10)\,\mathrm{P}(\operatorname{sign} Y \ne m) + x\,\mathrm{P}(\operatorname{sign} Y = m) \\ &= x + 10\,\mathrm{P}(\operatorname{sign} Y \ne m). \end{aligned}$$

As Y is a Normal$(mx, 1)$ random variable, the event $\operatorname{sign} Y \ne m$ occurs when Y is x standard deviations smaller than its mean for $m = 1$, or x standard deviations larger than its mean for $m = -1$. In either case,

$$\mathrm{P}(\operatorname{sign} Y \ne m) = \mathrm{P}(N \ge x) = 1 - \mathrm{P}(N \le x),$$

where N is a Normal$(0, 1)$ random variable. We want to find x that minimizes the expression

$$f(x) = \mathrm{E}[X] = x + 10(1 - \mathrm{P}(N \le x)).$$

The extremal points of f can occur at zero or at some $x \in [0, \infty)$ such that $f'(x) = 0$. As

$$f'(x) = 1 - 10\frac{d}{dx}\,\mathrm{P}(N \le x) = 1 - 10 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2},$$

the only such x is $x = \sqrt{2\ln(10/\sqrt{2\pi})} \approx 1.664$. As $\lim_{x\to\infty} f(x) = \infty$, x must be the minimum.