1. There are 5 red balls, 5 blue balls, and 5 green balls in a bin. You draw two balls from a bin. What is the probability that

   (a) Ball 2 is red?
   (b) Ball 2 is red given that ball 1 is red, if the balls are drawn *with* replacement?
   (c) Ball 2 is red given that ball 1 is red, if the balls are drawn *without* replacement?
   (d) Ball 2 is not blue given that ball 1 is red, if the balls are drawn *without* replacement?
   (e) Ball 2 is red given that ball 1 is not blue, if the balls are drawn *without* replacement?

   Solution: Let R_1 be the event "ball 1 is red" and B_1, G_1, R_2 be defined similarly,

   (a) $\mathrm{P}(R_2) = 1/3$ as the outcomes are equally likely.
   (b) $\mathrm{P}(R_2|R_1) = \mathrm{P}(R_2) = 1/3$. Owing to the replacement the color of the first ball does not affect the color of the second ball.
   (c) $\mathrm{P}(R_2|R_1) = 2/7$ because after the first ball is drawn there are 4 red balls to choose out of 14.
   (d) $\mathrm{P}(B_2^c|R_1) = 9/14$ because after the first ball is drawn there are 4 non-blue balls to choose out of 14.
   (e) The conditioned sample space given B_1^c is partitioned by the events R_1 and G_1. By the total probability theorem,

   $$\mathrm{P}(R_2|B_1^c) = \mathrm{P}(R_2|R_1)\,\mathrm{P}(R_1|B_1^c) + \mathrm{P}(R_2|G_1)\,\mathrm{P}(G_1|B_1^c) = \frac{2}{7}\cdot\frac{1}{2} + \frac{5}{14}\cdot\frac{1}{2} = \frac{9}{28}.$$

2. Roll a 6-sided dice three times. Let E_{12} be the event that the first face is *the same as* the second face. Define E_{23} and E_{13} analogously. Determine which of the following statements are true:

   (a) Any two of the three events E_{12}, E_{23}, E_{13} are independent.
   (b) E_{12}, E_{23} and E_{13} are independent.
   (c) E_{12} and E_{23} are independent conditioned on E_{13}.

   Solution: Our sample space will consist of all triples of possible faces (a, b, c) where a, b, and c are numbers between 1 and 6. We assume equally likely outcomes, so all triples occur with probability 6^{-3}. The probability for E_{12}, E_{23} and E_{13} are $\mathrm{P}(E_{12}) = \mathrm{P}(E_{23}) = \mathrm{P}(E_{13}) = 30/36$ by counting.

   (a) **True.** The intersection of E_{12} and E_{13} all three faces are equal. There are 6 outcomes, each occurring with probability 1/36, so

   $$P(E_{12} \cap E_{13}) = P(E_{12} \cap E_{23}) = P(E_{13} \cap E_{23}) = 6^{-2}.$$

   On the other hand, probability that any two of them are the same is

   $$P(E_{12}) = P(E_{23}) = P(E_{13}) = 6 \cdot \frac{1}{6^2} = 6^{-1}.$$

   Since $P(E_{12} \cap E_{23}) = 6^{-2} = P(E_{12}) \cdot P(E_{23})$, the two events E_{12} and E_{23} are independent, and similarly for the other two pairs.

(b) **False.** $E_{12} \cap E_{23} \cap E_{13}$ is also the event that all three faces are the same, so $\mathrm{P}(E_{12} \cap E_{23} \cap E_{13}) = 6^{-2}$. On the other hand $P(E_{12}) \cdot P(E_{23}) \cdot P(E_{13}) = 6^{-3}$ so the three events are not independent.

(c) **False.** Conditional independence holds when

$$\mathrm{P}(E_{23}|E_{13}) = \mathrm{P}(E_{23}|E_{13} \cap E_{12}).$$

The probability on the left is the ratio of the probabilities of $E_{23} \cap E_{13}$ and E_{13}, so it equals 6^{-1}. The probability on the right equals one, because if the first face and the second face are the same and the first face and the third face are the same, the first face and the third face must be also the same.

3. If Alice flips 10 coins and Bob flips 9 coins, what is the probability that Alice gets more heads than Bob? (Hint: Use conditioning.)

Solution: The sample space consists of all sequences of 19 `H`s and `T`s, where the first 9 elements in the sequence denote the outcomes of Bob's flips and the next 10 elements denote the outcomes of Alice's flips. We assume equally likely outcomes.

Let E be the event that Alice gets more heads than Bob. After Alice and Bob have both flipped their first nine coins, there are three possibilities: Alice gets more heads, Bob gets more heads, or there is a tie: Alice and Bob get the same number of heads. Let A, B and T denote those three events respectively. The events A, B, and T partition the sample space. By the rule of average conditional probabilities, we have

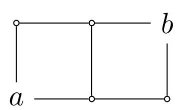
$$P(E) = P(E \mid A)P(A) + P(E \mid B)P(B) + P(E \mid T)P(T).$$

We now calculate the quantities on the right hand side. Given that event A happens, event E happens with certainty, so $P(E \mid A) = 1$. Similarly, given that event B happens, event E is impossible, so $P(E \mid B) = 0$. Given that event T happens, the game is decided by Alice's last coin toss. Since the event of Alice getting a head in her last toss is independent of the outcomes of the previous tosses, $P(E \mid T) = 1/2$. Therefore

$$P(E) = P(A) + \frac{1}{2}P(T) = P(A) + \frac{1}{2}(1 - P(A) - P(B)) = \frac{1}{2} + \frac{1}{2}P(A) - \frac{1}{2}P(B).$$

By symmetry, $P(A) = P(B)$, and so $P(E) = 1/2$.

4. Computers a and b are linked through seven cables as in the picture. Each cable fails with probability 10% independently of the others. Let C be the event "there is a connection between a and b" and F be the event "the middle vertical cable fails".



(a) What is the probability of C given F?

Solution: Let T and B be the events that a connects to b via the top and bottom paths respectively. Conditioned on F, events T and B are independent so

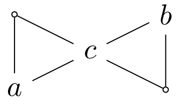
$$\mathrm{P}(C|F) = \mathrm{P}(T \cup B|F) = 1 - \mathrm{P}(T^c \cap B^c|F) = 1 - \mathrm{P}(T^c|F)\,\mathrm{P}(B^c|F).$$

Both T and B are independent of F, so by the algebra of independent events, $\mathrm{P}(T^c|F) = \mathrm{P}(T^c)$ and $\mathrm{P}(B^c|F) = \mathrm{P}(B^c)$ and we get that

$$\mathrm{P}(C|F) = 1 - \mathrm{P}(T^c)\,\mathrm{P}(B^c) = 1 - (1 - 0.9^3)^2.$$

(b) What is the probability of C given F^c?

Solution: Let c be the top middle node. Conditioned on F^c, we can contract the top and bottom single nodes and picture the network like this:



Let L and R be the events "there is a connection from a to c" and "there is a connection from c to b", respectively. They are independent so

$$\mathrm{P}(C|F^c) = P(L \cap R|F^c) = P(L|F^c)P(R|F^c) = (1 - P(L^c|F^c))(1 - P(R^c|F^c)).$$

The complement of L (given F^c) happens when both of the connections from a to c fail. Since they are independent,

$$\mathrm{P}(L^c|F^c) = 0.1 \times (1 - 0.9^2).$$

By symmetry, $\mathrm{P}(R^c|F^c) = 0.1 \times (1 - 0.9^2)$, and so

$$\mathrm{P}(C|F^c) = (1 - 0.1 \cdot (1 - 0.9)^2)^2.$$

(c) What is the probability of C?

Solution: By the total probability theorem,

$$P(E) = 0.1 \times (1 - (1 - 0.9^3)^2) + 0.9 \times (1 - 0.1 \times (1 - 0.9^2))^2 \approx 0.959.$$