

1. Alice, Bob, and Charlie hold a lucky draw for two tickets to a concert with the following odds:

- The probability that Alice gets one of the tickets is 60%.
- The probability that Bob gets one of the tickets is 70%.

What is the probability that Alice and Bob both get tickets?

Solution: The sample space consists of the three outcomes $\{ab, ac, bc\}$, where ab represents Alice and Bob getting the tickets, and so on. Denote their probabilities by p_{ab} , p_{ac} , and p_{bc} . The event “Alice gets one of the tickets” is $\{ab, ac\}$ so $p_{ab} + p_{ac} = 0.6$. Similarly $p_{ab} + p_{bc} = 0.7$. Since the probabilities must add up to one,

$$p_{ab} = (p_{ab} + p_{ac}) + (p_{ab} + p_{bc}) - 1 = 0.6 + 0.7 - 1 = 0.3.$$

Alternative solution: Let A , B , and C be the events “Alice gets a ticket” and so on. By the axioms the complementary events have probabilities $P(A^c) = 0.4$ and $P(B^c) = 0.3$. The events A^c , B^c , and C^c partition the sample space (they are disjoint and exactly one of the three must occur) so their probabilities must add up to one. Therefore $P(C^c) = 1 - 0.4 - 0.3 = 0.3$. The event C^c happens exactly when Alice and Bob both get tickets, so the desired probability is 30%.

2. A machine produces parts that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it. What is the quality of the parts that make it through the inspection machine and get shipped?

Solution: The sample space consists of the three possible outcomes described in the statement. Let G , SD , OD be the events describing each of these outcomes. The probability model is $P(G) = 0.90$, $P(SD) = 0.02$, and $P(OD) = 0.08$. We want to compute the conditional probability that a part is good given that it passed the inspection machine (i.e., it is not obviously defective), which is

$$P(G|OD^c) = \frac{P(G \cap OD^c)}{P(OD^c)} = \frac{P(G)}{1 - P(OD)} = \frac{0.9}{1 - 0.08} \approx 0.978.$$

As the events G and SD partition the conditional sample space, from the axioms of probability we get that $P(SD|OD^c) = 1 - P(G|OD^c) \approx 0.022$.

3. You flip a fair coin five times independently. Let H_1 be the event that the first flip is a head and M be the event that a majority (at least 3 out of 5) of the flips are heads. Calculate (a) $P(M)$; (b) $P(M|H_1)$; (c) $P(H_1|M)$.

Solution: The sample space is the set that contains all outcomes for flipping a coin five times, which has the size 2^5 .

- (a) The size of event M is $\binom{5}{3} + \binom{5}{4} + \binom{5}{5}$. Therefore, the probability is $P(M) = (\binom{5}{3} + \binom{5}{4} + \binom{5}{5})/2^5 = 0.5$.

- (b) The conditional probability is $P(M|H_1) = P(M \cap H_1)/P(H_1)$. First, we get the probability of H_1 , which is $1/2$. The event $M \cap H_1$ is equivalent to the event that a majority (at least 2 out of 4) of the flips are heads and the first flip is head, and by the same calculation as (a), we get $P(M \cap H_1) = ((\binom{4}{2}) + \binom{4}{3})/2^5 = 0.34375$. Therefore, the conditional probability is $P(M|H_1) = P(M \cap H_1)/P(H_1) = 0.6875$.
- (c) The conditional probability is $P(H_1|M) = P(H_1 \cap M)/P(M) = 0.34375/0.5 = 0.6875$.
4. A bin contains 3 white balls and 5 black balls. Alice and Bob take turns drawing balls from the bin without replacement until a white ball is drawn. Assuming Alice goes first, what is the probability that she gets the white ball?

Solution: Our sample space will consist of all $\binom{8}{3}$ arrangements of 3 white balls and 5 black balls. The first ball in the arrangement is drawn by Alice, the second by Bob (if necessary), the third by Alice (if necessary), and so on. We assume equally likely outcomes.

Let E_i be the event that Alice selects the white ball in the i -th turn, and therefore all balls drawn in previous turns are black. This event consists of all arrangements that start with $i - 1$ blacks followed by a white. We are interested in the probability of the event E that Alice selects the white ball at *some* turn, that is:

$$E = E_1 \cup E_3 \cup E_5.$$

(E_7 and so on are empty because there are not enough black balls.) The three events are pairwise disjoint so

$$|E| = |E_1| + |E_3| + |E_5|.$$

Since the arrangement of the first i balls is fixed and contains exactly one white, the size of E_i is the number of arrangements of the remaining $8 - i$ balls out of which two are white. Therefore $|E_i| = \binom{8-i}{2}$, and by the equally likely outcomes formula

$$P(E) = \frac{\binom{7}{2} + \binom{5}{2} + \binom{3}{2}}{\binom{8}{3}} \approx 0.607.$$