Quiz 8 solution

Five boy-girl couples arrive in a restaurant. The ten guests take up random places at a round table. Let N be the number of couples that end up in adjacent seats. What is the variance of N?

Solution: Let N_i be the indicator random variable for the event that the i-th couple is seated together. Then $N = N_1 + N_2 + N_3 + N_4 + N_5$. By the total variance formula,

$$\mathrm{Var}[N] = \sum_{i=1}^{5} \mathrm{Var}[N_i] + \sum_{i \neq j} \mathrm{Cov}[N_i, N_j]. \tag{1}$$

Since N_i is an indicator random variable, $\mathrm{Var}[N_i] = p - p^2$, where p is the probability that the i-th couple is seated together. Once the boy's seat is fixed there are two out of nine equally likely choices for the girl's seat that place them next to one another, so $p = 2/9$ and

$$\mathrm{Var}[N_i] = 2/9 - (2/9)^2 = 14/81.$$

The covariance terms are equal to

$$\mathrm{Cov}[N_i, N_j] = \mathrm{E}[N_i, N_j] - \mathrm{E}[N_i]\,\mathrm{E}[N_j] = \mathrm{P}(N_i = 1, N_j = 1) - p^2 = \mathrm{P}(N_j = 1 | N_i = 1) \cdot p - p^2.$$

Given that the i-th couple is seated together, there are $8 \cdot 7$ possible seatings for the j-th couple, out of which they sit together in $2 \cdot 7$ of them. By the equally likely outcomes formula,

$$\mathrm{P}(N_j = 1 | N_i = 1) = \frac{2 \cdot 7}{8 \cdot 7} = \frac{1}{4}$$

and so

$$\mathrm{Cov}[N_i, N_j] = \frac{1}{4} \cdot \frac{2}{9} - \left(\frac{2}{9}\right)^2 = \frac{1}{18} - \frac{4}{81} = \frac{1}{162}.$$

Therefore

$$\mathrm{Var}[N] = 5 \cdot \frac{14}{81} + 20 \cdot \frac{1}{162} = \frac{80}{81} \approx 0.988.$$