Alice is about to take two laps in the CUHK swimming pool. The time of her first lap is F minutes, where F is an Exponential(1) random variable. The time of her second lap is S minutes, where S is an Exponential(F) random variable. What is the probability that she completes her second lap within one minute?

Solution: By the total probability theorem

$$\mathrm{P}(S \leq 1) = \int_0^\infty \mathrm{P}(S \leq 1 \mid F = f) f_F(f) df,$$

where f_F is the PDF of an Exponential(1) random variable, and $\mathrm{P}(S \leq 1 \mid F = f)$ is the CDF of an Exponential(f) random variable evaluated at 1. Therefore

$$\mathrm{P}(S \leq 1) = \int_0^\infty \big(1 - e^{-f}\big) e^{-f} df = \int_0^\infty \big(e^{-f} - e^{-2f}\big) df = \big(-e^{-f} + \tfrac{1}{2} e^{-2f}\big)\big|_0^\infty = \frac{1}{2}.$$

As an aside, an Exponential(1) random variable is not a realistic model for the time of a lap in minutes as it predicts that, say, Alice completed her first lap within 5 seconds with probability $1 - e^{-5/60}$ which is about 8%!