ENGG 2760A / ESTR 2018: Probability for Engineers — Quiz 3 solution
The Chinese University of Hong Kong, Fall 2020

Alice draws cards one by one from a shuffled 52-card deck. Find the PMF of the turn T at which she has drawn the fourth (and last) ace.

Solution: The sample space consists of all $\binom{58}{4}$ arrangements of four (indistinguishable) aces and 48 other cards under equally likely outcomes. The event $T = t$ occurs when the first $t-1$ cards contain 3 aces and the t-th card is an ace. By the product rule there are $\binom{t-1}{3}$ such arrangements. By the equally likely outcomes formula

$$P(T = t) = \frac{\binom{t-1}{3}}{\binom{52}{4}} = \frac{4 \cdot (t-1)(t-2)(t-3)}{52 \cdot 51 \cdot 50 \cdot 49} \tag{1}$$

for $1 \leq t \leq 52$.

Alternative solution: The event $T = t$ occurs if no ace is drawn in turns $t+1$ up to 52 and an ace is drawn in turn t. Let A_i be the event that an ace is drawn in turn i. By the multiplication rule

$$\begin{aligned}
P(T = t) &= P(A_t \cap A_{t+1}^c \cap \cdots \cap A_{52}^c) \\
&= P(A_{52}^c) \cdot P(A_{51}^c | A_{52}^c) \cdots P(A_{t+1}^c | A_{t+2}^c \cap \cdots \cap A_{52}^c) \cdot P(A_t | A_{t+1}^c \cap \cdots \cap A_{52}^c) \\
&= \frac{48}{52} \cdot \frac{47}{51} \cdots \frac{t-3}{t+1} \cdot \frac{4}{t}
\end{aligned}$$

after cancellation this expression reduces to (1).