1. Let A and B be arbitrary events. Which of the following is true? If you answer yes, prove it using the axioms of probability. If you answer no, provide a counterexample.

   (a) $\mathrm{P}(A|B) + \mathrm{P}(A|B^c) = 1$.

   Solution: No. If B is the event of a fair coin flipping heads and A is the event of the coin flipping heads or tails then $\mathrm{P}(A|B) = 1$ and $\mathrm{P}(A|B^c) = 1$.

   (b) $\mathrm{P}(A \cap B|A \cup B) \leq \mathrm{P}(A|B)$.

   Solution: Yes, because $\mathrm{P}(A \cup B) \geq \mathrm{P}(B)$ and so

$$\mathrm{P}(A \cap B|A \cup B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A \cup B)} \leq \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \mathrm{P}(A|B).$$

2. n independent random numbers are sampled uniformly from the interval $[0, 1]$.

   (a) If $n = 10$, what is the probability that exactly 4 of them are greater than 0.7?

   Solution: Let N be the number of such random numbers greater than 0.7. Then N is a binomial random variable with $n = 10$ samples and success probability $p = 3$, so

$$\mathrm{P}(X = 4) = \binom{10}{4} 0.3^4 (1 - 0.3)^{10-4} \approx 0.200.$$

   (b) If $n = 50$, use the Central Limit Theorem to estimate the probability that their sum is between 20 and 25 (inclusive).

   Solution: Let X_i denote the value of the i-th random number ($i = 1, \ldots, n$). Then $X_1, \ldots, X_n$ are independent random variables with mean 1/2 and variance 1/12. Let $X = X_1 + \cdots + X_n$. Then $\mathrm{E}[X] = 25$ and $\mathrm{Var}[X] = 50/12$. By the Central Limit Theorem, the CDF of X can be approximated by the CDF of a Normal$(25, \sqrt{50/12})$ random variable $\tilde{N}$. Normalizing $\tilde{N} = 25 + N \cdot \sqrt{50/12}$,

$$\begin{aligned}
\mathrm{P}(20 \leq X \leq 25) &\approx \mathrm{P}(20 \leq \tilde{N} \leq 25) \\
&= \mathrm{P}(20 \leq 25 + N \cdot \sqrt{50/12} \leq 25) \\
&= \mathrm{P}(-5/\sqrt{50/12} \leq N \leq 0) \\
&\approx \mathrm{P}(-2.45 \leq N \leq 0) \\
&= F_N(0) - F_N(-2.45) \\
&\approx 0.5 - 0.0071 \\
&\approx 0.4929.
\end{aligned}$$

3. Companies A and B produce lightbulbs. Their lifetimes are exponential random variables with mean 2 years for company A and 1 year for company B.

   (a) A shop sources 3/4 of its lightbulbs from company A and the remaining 1/4 from company B. If a random lifebulb from the shop survived for 2 years, how likely is it to have been produced by company B?

   Solution: Let X be the lifetime of the lightbulb, and A and B be the (complementary) events that the respective company produced it. Then $\mathrm{P}(X \geq t|A) = e^{-t/2}$ and $\mathrm{P}(X \geq t|B) = e^{-t}$. By the total probability theorem,

$$\mathrm{P}(X \geq 2) = \mathrm{P}(X \geq 2|A)\,\mathrm{P}(A) + \mathrm{P}(X \geq 2|B)\,\mathrm{P}(B) = e^{-1} \cdot \frac{3}{4} + e^{-2} \cdot \frac{1}{4} \approx 0.310.$$

   and by Bayes' rule

$$\mathrm{P}(B|X \geq 2) = \frac{\mathrm{P}(X \geq 2|B)\,\mathrm{P}(B)}{\mathrm{P}(X \geq 2)} = \frac{e^{-2} \cdot 1/4}{e^{-1} \cdot 3/4 + e^{-2} \cdot 1/4} \approx 0.109.$$

   (b) What is the probability that a lightbulb produced by company B outlasts one produced by company A? Assume their lifetimes are independent.

   Solution: Let X and Y be their respective lifetimes. The PDF of X is $f_X(x) = \frac{1}{2}e^{-x/2}$. By the total probability theorem,

$$\begin{aligned}
\mathrm{P}(Y > X) &= \int_0^\infty \mathrm{P}(Y > X|X = x) f_X(x)dx \\
&= \int_0^\infty \mathrm{P}(Y > x) f_X(x)dx \\
&= \int_0^\infty e^{-x} \cdot \tfrac{1}{2}e^{-x/2}dx \\
&= \frac{1}{3}.
\end{aligned}$$

   Alternative solution: If we divide each year into n equal intervals and flip independent coins of probabilities $1/2n$ and $1/n$ for the failure of each bulb in each interval, then X and Y are the times of the first failure in the limit as n goes to infinity. For a fixed n, let A_n be the event that ligthbulb A failed in the interval in which the first lightbulb failure occurred. Then A_n has the same probability as the event that the first coin came up heads given that at least one did, namely

$$\mathrm{P}(A_n) = \frac{1/2n}{1 - (1 - 1/2n)(1 - 1/n)} = \frac{1/2n}{1/2n + 1/n - (1/2n)(1/n)} = \frac{1}{3 + 1/n}.$$

   As n tends to infinity, $\mathrm{P}(A_n)$ tends to $1/3$ so $\mathrm{P}(Y > X) = 1/3$.

4. Alice takes T hours to travel to Bob's house, where T is a random variable with PDF

$$f_T(t) = \begin{cases} 1/t^2, & \text{when } t \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the CDF (cumulative distribution function) $F_T(t) = \mathrm{P}(T \leq t)$.

   Solution: $F_T(t)$ is zero when $t < 1$. When $t \geq 1$,

$$F_T(t) = \int_1^t \frac{1}{u^2}\, du = -\frac{1}{u}\bigg|_1^t = 1 - \frac{1}{t}.$$

   (b) The distance between Alice's and Bob's house is one mile so that Alice travels at a speed $V = 1/T$ miles per hour. What is Alice's expected speed $\mathrm{E}[V]$?

   Solution: The CDF $F_V(v)$ of V is zero when $v \leq 0$. If $v > 0$,

$$\mathrm{P}(V \leq v) = \mathrm{P}(1/T \leq v) = \mathrm{P}(T \geq 1/v) = 1 - F_T(1/v) = \begin{cases} v, & \text{if } 0 \leq v \leq 1, \\ 1, & \text{if } v \geq 1. \end{cases}$$

   This is the CDF of a Uniform$(0, 1)$ random variable, so $\mathrm{E}[V] = 1/2$.

5. A group of 10 boys and 10 girls is randomly divided into 5 teams A, B, C, D, E with 4 children per team.

   (a) What is the probability that all children in team A are of the same gender?

   Solution: By the multiplication rule, this probability is $p = 9/19 \cdot 8/18 \cdot 7/17 \approx 0.087$.

   (b) Is the probability that all teams are of mixed gender more than 50% or not? Justify your answer.

   Solution: It is. Let S be the number of same gender teams. By linearity of expectation, $\mathrm{E}[S] = 5p \approx 0.433$. By Markov's inequality, $\mathrm{P}(S \geq 1) \leq \mathrm{E}[S]$, so the complementary event $S = 0$ occurs with probability at least $1 - 0.433 > 0.5$.