ENGG 2760A / ESTR 2018: Probability for Engineers

Tutorial 5

1. Quiz 3
2. Quiz 4
3. Homework 6

Quiz 3:

Alice draws cards one by one from a shuffled 52-card deck. Find the PMF of the turn T at which she has drawn the fourth (and last) ace.

$P(T=52)=1$

$P(T=t)=P(\underset{\uparrow}{X}\leq t)$ CDF

draw the last ace at turn X

Quiz 4:

Eight boys and eight girls are randomly seated at a round table. What is the expected number of boys that are seated between two girls?

32/15

Linearity of expectation

Indicator random variable

$$X_i = \begin{cases} 1 & \text{if } i\text{th boy seated between 2 girls} \\ 0 & \text{o/w} \end{cases}$$

$$X = X_1 + \cdots + X_8$$

$$\Rightarrow E[X] = E[X_1] + \cdots + E[X_8]$$

girls

$$E[X_i] = P(X_i = 1) = \frac{\binom{8}{2}}{\binom{15}{2}} = \frac{8\times7}{15\times14}$$

$$E[X] = 8 \times \frac{8\times7}{15\times14} = \frac{32}{15}$$

HW6

Q1(a): $P_{M|N}(M=m \mid N=n) = \dfrac{P_{M,N}(m,n)}{P_N(n)}$ ← joint ← marginal

(b) If M, N are independent.

$P_{M,N}(m,n) = P_M(m) \times P_N(n)$

not independent

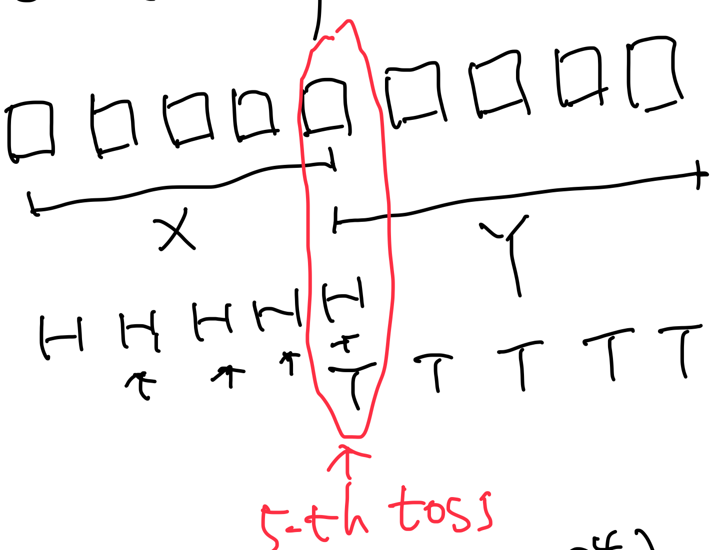
find a contradiction

(c) $\underbrace{E[M \mid N<2]}$ $= \sum_m m P_m(m \mid N<2) = 6$

$$\Downarrow$$

$$\frac{P_{m,n}(m,1) + P_{m,n}(m,0)}{P_N(n<2)}$$

Q2(a) 9 times



$P(X=4) = \frac{1}{2^5}$

$P(Y=4) = \frac{1}{2^5}$

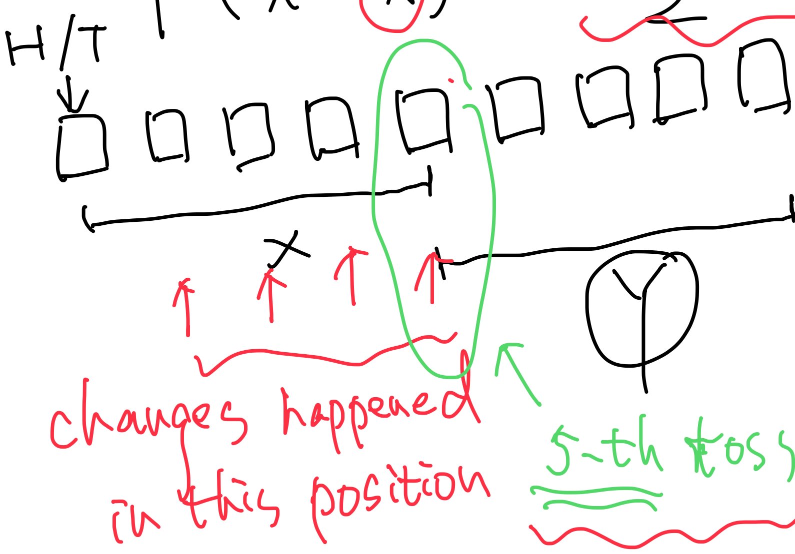
dependent

$\Leftarrow P_{X,Y}(4,4) = 0$

$\neq P_X(4) \cdot P_Y(4) = \frac{1}{2^{10}}$

(b) $P(X=x) = \frac{C^4_x \times 2}{2^5}$

$P(Y=y) = \frac{C^4_y \times 2}{2^5}$



5-th position is H

$P_{X,Y}(x,y) = P_{X,Y}(x,y \mid H)\,P(H) + P_{X,Y}(x,y \mid T)\,P(T)$

$= \frac{C^4_x\, C^4_y}{2^8} \times \frac{1}{2} + \frac{C^4_x\, C^4_y}{2^8} \times \frac{1}{2}$

$= \frac{C^4_x\, C^4_y}{2^8} = P_X(x) P_Y(y) = \frac{C^4_x\, C^4_y}{2^8}$

(c) $P(X=0)$ first 5 tosses, not TH

$P(Y=0)$ last 5 tosses, not HT

$$P(X=0) \xleftrightarrow{\text{chang } T \Leftrightarrow H} P(Y=0)$$

T T T T T
H T T T T
H H T T T
H H H T T
H H H H T
H H H H H

H H H H H
T H H H H
⋮
T T T T H
T T T T T

$$\left(\frac{6}{2^5} \times \frac{6}{2^5}\right) = \frac{36}{2^{10}} = \frac{9}{2^8}$$

$$P_{X,Y}(0,0) = P_{X,Y}(0,0 \mid \underset{\substack{\uparrow \\ \text{5-th}}}{H})\,P(H) + P_{X,Y}(0,0 \mid T)\,P(T)$$

$$= \frac{1}{2^8} \times \frac{1}{2} + \frac{5 \times 5}{2^8} \times \frac{1}{2} = \frac{26}{2^8}$$

$$\neq P_X(0)\,P_Y(0)$$

Q3: $37 = 18R + 18B + 1G$

(a)

$$E[X] = E[X_1] + E[X_2] + E[X_3] = 3 \times \left(2 \times \frac{18}{37} + 0 \times \frac{19}{37}\right) = \frac{108}{37}$$

$$Var[X] = \sum_{i=1}^{3} Var[X_i] = \sum_{i=1}^{3} \left[E[X_i^2] - (E[X_i])^2\right] = 2.997$$

$$\sigma(X) = \sqrt{Var(X)} \approx 1.733$$

(b) $3 \xrightarrow{1\,win} 6 \xrightarrow{2\,win} 12 \xrightarrow{3\,win} 24$

$$E[X] = 24 \times \left(\frac{18}{37}\right)^3 \approx 2.763$$

$$Var[X] = E[X^2] - (E[X])^2 \approx 58.68$$

$$\sigma(X) = 7.669$$

Q(4)

(a)

$H \underbrace{HHT}_{X_1} \underbrace{TTTH}_{X_2}$ ← we continue to flip X_2 time until get the result different from the last result of X_1

After we flip the coin at first time, we continue to flip X_1 times until the result if different from the first time

$X_1 \sim Geometric(\underset{=}{p})$, $p = \frac{1}{2}$

$X_2 \sim$ ↑

$$E[X] = 1 + E[X_1] + E[X_2]$$

$$= 1 + 2 + 2 = 5$$

(b)

$$\underbrace{1\,1\,3}_{X_1}\ \underbrace{3\,3\,2}_{X_2}$$

2 or 3

$$E[X] = 1 + \underbrace{E[X_1]} + E[X_2] = 1 + \frac{3}{2} + 3 = \frac{11}{2}$$

$X_1 \sim \text{Geometric}\left(\frac{2}{3}\right)$

$\underline{X_2} \sim \text{Geometric}\left(\frac{1}{3}\right)$