Side Talk: More on Big-O

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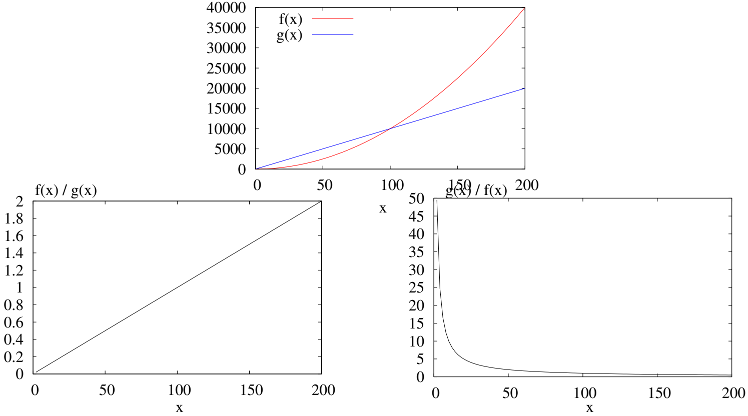
In the class, we have learned that, intuitively, $f(n) = O(g(n))$ means "function $f(n)$ grows asymptotically no faster than function $g(n)$". In the next few slides, we will reinforce this understanding from a graphical point of view.

Quadratic vs. Linear

$f(n) = n^2$ and $g(n) = 100n$.

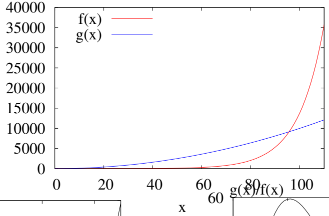


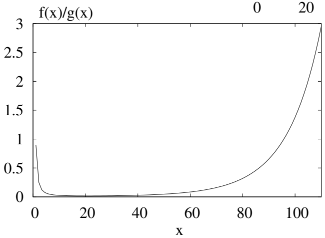
So we know $g(n) = O(f(n))$.

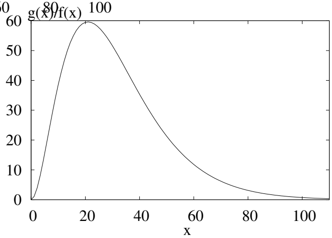
Note that we can scale up $f(x)$ a constant times to make the red line always above the blue line.

Exponential vs. Quadratic

$f(n) = 1.1^n$ and $g(n) = n^2$.





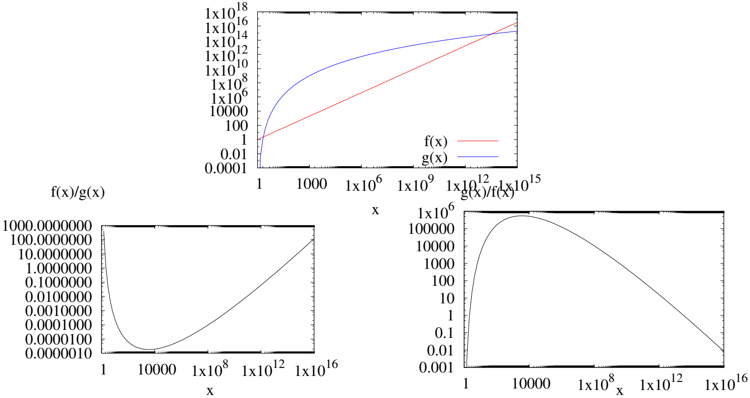


So we know $g(n) = O(f(n))$.

Note that we can scale up $f(x)$ a constant times to make the red line always above the blue line.

Polynomial vs. Poly-Logarithmic

$f(n) = n^{1.1}$ and $g(n) = (\log_2 n)^9$.

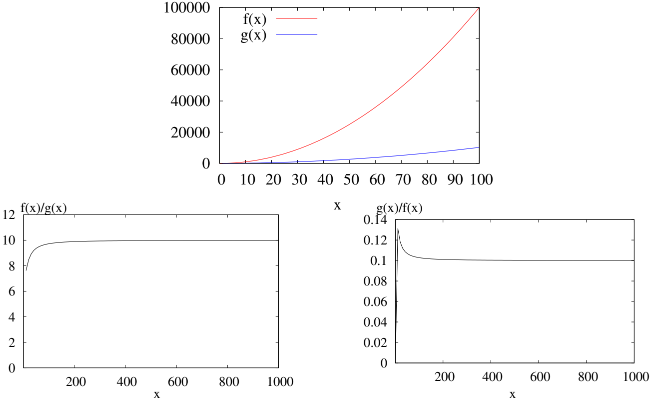


So we know $g(n) = O(f(n))$.

Note that we can scale up $f(x)$ a constant times to make the red line always above the blue line.

An Example of Θ

$f(n) = 10n^2$ and $g(n) = n^2 - \sqrt{n} + (\log_2 n)^3$.



So we know $g(n) = \Theta(f(n))$.

Clearly the blue line is always below the red line. But we can also scale up $g(x)$ a constant times to make the blue line always above the red line (figure this out from the left figure of the 2nd row).

Our final words concern the definition of big-O. Recall that our "official" definition of $f(n) = O(g(n))$ is:

> There is a constant $c_1 > 0$ such that $f(n) \le c_1 \cdot g(n)$ holds for all n at least a constant c_2.

In the lecture, we also mentioned that $f(n) = O(g(n))$ when $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is at most some constant c. This provides an alternative approach to prove the big-O.

However, it must be emphasized that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \le c$ is only a sufficient condition of big-O, but not a necessary condition. Why? Because it is possible that $f(n) = O(g(n))$, and yet, $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist! We will see an example in the next slide.

Consider $f(n) = 2^n$. Define $g(n)$ as:

- $g(n) = 2^n$ if n is even;
- $g(n) = 2^{n-1}$ otherwise.

Since $f(n) \leq 2g(n)$ holds for all $n \geq 1$, it holds that $f(n) = O(g(n))$.

However, $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ does not exist, because it keeps jumping between 1 and 2 as n increases!