Dynamic Arrays and Amortized Analysis

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To create an array, you need to specify a size, i.e., how many elements you can store in the array. Increasing the size is expensive because it means creating a new array and moving all the elements over.

This lecture will discuss clever tricks to change the array size efficiently! Our discussion introduces the method of **amortized analysis**.

Dynamic Array Problem

Let S be a collection of integers (not necessarily distinct). S is empty in the beginning. Integers are then added to S one by one with **insertions**.

Let n be the number of elements in S currently. We want to maintain an array A satisfying:

1. A has length $O(n)$.
2. For each $i \in [1, n]$, $A[i] = x$ if x is the i-th integer added to S.

The above requirements need to be satisfied after every insertion.

Naive Algorithm

Perform insert(e) (which inserts an integer e to S) as follows:

- If $n = 0$, set n to 1 and initialize A to have length 1 to store e.
- Otherwise ($n \geq 1$):
  - Increase n by 1.
  - Initialize an array A' of length n.
  - Copy all the $n - 1$ elements of A to A'.
  - Set $A'[n] = e$.
  - Destroy A and replace it with A'.

This algorithm spends $O(n)$ time on the n-th insertion. Altogether, it takes $O(n^2)$ time to do n insertions.

We will reduce the time of inserting n elements dramatically to $O(n)$. Our array A may have a length up to $2n$.

A Better Algorithm

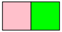
A is **full** if its cells are all filled.

Perform insert(e) as follows:

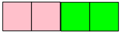
- If $n = 0$, set n to 1 and initialize A of length 2 to store just e itself.
- Otherwise (i.e., $n \geq 1$), append e to A and increase n by 1. If A is full:
  - Initialize an array A' of length $2n$.
  - Copy all the elements of A to A'.
  - Destroy A and replace it with A'.

Example

$n = 1$

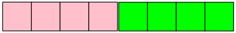


$n = 2$



$n = 3$

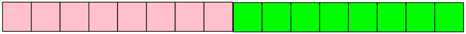
$n = 4$



$n = 5$

...

$n = 8$



Analysis

Cost of inserting the n-th element:

- if A is not full after the insertion, $O(1)$;
- otherwise, $O(n)$, i.e., the time of expanding A.

Analysis

Array expansions are infrequent:

- Initially, size 2.
- 1st expansion: size from 2 to 4.
- 2nd expansion: from 4 to 8.
- ...
- i-th expansion: from 2^i to 2^{i+1}.

After n insertions, the size of A is at most $2n$. Hence:

$$2^{i+1} \le 2n \quad \Rightarrow \quad i \le \log_2 n$$

that is, at most $\log_2 n$ expansions.

Analysis

The total cost of n insertions is bounded by:

$$\left(\sum_{i=1}^{n} O(1)\right) + \sum_{i=1}^{\log_2 n} O(2^i) \tag{1}$$

where

- the first term captures the $O(1)$ time compulsory for each insertion;
- the second term captures all the expansion cost.

(1) evaluates to $O(n)$.

We have shown that the total cost of n insertions is $O(n)$. In other words, each insertion entails $O(1)$ cost "on average". This does not mean that every insertion can be performed in $O(1)$ time. The cost of some insertions can reach $\Omega(n)$.

In general, if a data structure can process any n operations in $f(n)$ time, we say that it guarantees an **amortized cost** of $\frac{f(n)}{n}$ per operation.

The dynamic array guarantees $O(1)$ amortized cost per insertion.