ENGG1410-F Tutorial:

A Closer Look at Linear Systems with Infinite Solutions

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We learned about linear transformations. Today we will see an important application of this concept: finding all solutions to a linear system when there are infinitely many.

Let us warm up by discussing the projection of a set V of vectors. Take any V, e.g.:

$$[3, 0, 1, 2]$$
$$[6, 1, 0, 0]$$
$$[12, 1, 2, 4]$$
$$[6, 0, 2, 4]$$

The projection of V onto the, say, 2nd and 3rd components is the following set V' of vectors:

$$[0, 1]$$
$$[1, 0]$$
$$[1, 2]$$
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Can you give a very short proof of the following claim: the dimension of V is at least that of V'.

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Proof: The rank of a matrix is at least the rank of any sub-matrix. □

In general, let V be any (perhaps infinite) set of vectors. By taking the same components of the vectors in V, we get a projection of V, which is a set V' of vectors.

The dimension of V is at least the dimension of V'

We leave the simple proof to you (this is actually a problem in an exercise list on the course homepage).

Now we cut into our main topic: linear system with infinitely many solutions. Consider the following system:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Remark: This is another problem in the same exercise list.

The system can be transformed into:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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First set x_4, x_5 to any real numbers (i.e., they are unconstrained).
Then solve x_1, x_2, x_3 as:

$$\begin{aligned} x_1 &= -(x_4 + x_5) \\ x_2 &= -x_5 \\ x_3 &= -x_5. \end{aligned}$$

Now we ask the question: what is the dimension of V?

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Now we ask the question: what is the dimension of V?
Next, we show that the answer is 2, i.e., the number of variables **minus** the rank of the coefficient matrix!

Denote by V' the set of all vectors $\begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$.
Clearly, V' has dimension 2 (remember: x_4, x_5 are unconstrained).

$$\begin{aligned} x_1 &= -(x_4 + x_5) \\ x_2 &= -x_5 \\ x_3 &= -x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

That is, V can be obtained from V' through a linear transformation!

We know from the lecture that linear transformations do not increase the dimension! Therefore, the dimension of V is **at most** the dimension of V'. In other words, the dimension of V is at most 2.

V': the set of all vectors $\begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$.

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On the other hand, note that V' is the projection of V onto the 4-th and 5-th components. From our earlier discussion, we know that the dimension of V is **at least** the dimension of V'. In other words, the dimension of V is at least 2.

We now conclude that the dimension of V is precisely 2.