THE CHINESE UNIVERSITY OF HONG KONG

ENGG1410 (A-F) Linear Algebra and Vector Calculus for Engineers

Mid-term Examination

Instructions:

- This paper must be answered in **English**. The duration of this paper is **1.5 hours.**
- Attempt **ALL** questions. The full marks of this paper is **100 points.**
- Unless otherwise specified, numerical answers should be either **exact** or **correct to 3 significant figures.**
- **Do not take this paper away.** This paper has to be returned after the examination.

Question 1. Consider the following vectors, where a is a real number:

$$[1 \quad 8 \quad 6 \quad 4 \quad 0]$$

$$[8 \quad 5 \quad 7 \quad 6 \quad 9]$$

$$[4 \quad 6 \quad 8 \quad 7 \quad 1]$$

$$[5 \quad a \quad 5 \quad 3 \quad a+1]$$

(a) If these vectors are linearly dependent, what should be the value of a? (8 pts)

(b) Now assume that a takes a value such that these vectors are linearly independent.

Consider the following system of homogeneous equations

$$\begin{array}{llllll} x & +8y & +6z & +4u & & = 0 \\ 8x & +5y & +7z & +6u & +9w & = 0 \\ 4x & +6y & +8z & +7u & +w & = 0 \\ 5x & +ay & +5z & +3u & +(a+1)w & = 0 \end{array}$$

Is it possible that a nontrivial solution to the system exists, in which $w = 0$. What should be the value of a if such a solution exists? (8pts)

Question 2.

(a) Given the following two matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Compute the determinants for the following: $\det(\mathbf{A}), \det(\mathbf{B}), \det(\mathbf{A}^\mathbf{T}\mathbf{B})$. (6pts)

(b) Balance the following chemical reactions by setting up and solving a homogeneous linear system with respect to unknown variables x_1, x_2, x_3, x_4 that keeps the number of atoms of all kinds identical before and after the reaction. Please also describe the system's coefficient matrix's row and column vector space and dimensionality and the system's solution space dimensionality. Note that C is carbon, H is hydrogen, O is oxygen and Cl is chlorine.

$$x_1 \mathrm{CH_3CH_2OH} + x_2 \mathrm{Cl_2} \rightarrow x_3 \mathrm{CCl_3CHO} + x_4 \mathrm{HCl}$$

(10pts)

Question 3. Let

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$$

(a) Calculate $\mathbf{A}\mathbf{A}^\mathbf{T}$. (6pts)

(b) Using the result in part (a), or otherwise, find $\mathbf{A}^{-1}$. (6pts)

(c) Using the result in part (b), or otherwise, solve the following linear system:

$$\begin{array}{rcrcrcr} 2x_1 & - & 2x_2 & + & x_3 & = & 9 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 0 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 0 \end{array}$$

(4pts)

Question 4. Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix,

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

(16pts)

Question 5. Given a 2×2 matrix **A** with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and corresponding eigenvectors $\mathbf{e_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{e_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Determine the matrix **A**. (6pts)

(b) Given another matrix $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$. It is known that **B** has the same eigenvalues as matrix **A**. Determine the corresponding eigenvectors for **B**. (6pts)

(c) Are matrices **A** and **B** similar? If yes, describe how you would obtain a similarity transformation matrix **T** such that $\mathbf{T^{-1}AT} = \mathbf{B}$. (8pts)

Question 6. The trace of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is defined to be the sum of the entries lying on the main diagonal of **A**. That is $trace(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{k=1}^{n} a_{kk}$. Prove the following statements:

(a) For any $m \times n$ matrix **B** and $n \times m$ matrix **C**, $trace(\mathbf{BC}) = trace(\mathbf{CB})$. (8pts)

(b) Suppose that **B** is an $m \times n$ matrix. If $trace(\mathbf{B^T B}) = 0$, then **B** must be the zero matrix (i.e., $\mathbf{B} = \mathbf{0}$) (8pts)