Lecture Notes: Dot Product and Cross Product

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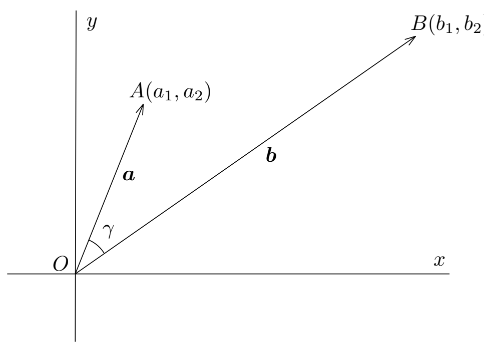
1 Angle between Two Vectors

Definition 1. *Given two non-zero vectors* $\boldsymbol{a} = [a_1, ..., a_d]$ *and* $\boldsymbol{b} = [b_1, ..., b_d]$*, we define their* **angle** *as the smaller angle*[1] *between the lines* $\ell_{\boldsymbol{a}}$ *and* $\ell_{\boldsymbol{b}}$*, where* $\ell_{\boldsymbol{a}}$ *is the line passing the origin and the point* $(a_1, ..., a_d)$*, and similarly* $\ell_{\boldsymbol{b}}$ *is the line passing the origin and the point* $(b_1, ..., b_d)$*.*

The figure below shows an example in two-dimensional space. Points A and B have coordinates (a_1, a_2) and (b_1, b_2), respectively. Thus, $\boldsymbol{a}$ is the vector defined by the directed segment $\overrightarrow{OA}$, and $\boldsymbol{b}$ is the vector defined by the directed segment $\overrightarrow{OB}$. The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is the angle γ as indicated in the figure between the two directed segments. Note that the angle of two vectors always falls between 0 and 180 degrees.



We say that vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are *orthogonal* if their angle is $90°$.

2 Dot Product Revisited

Recall that given two vectors $\boldsymbol{a} = [a_1, ..., a_d]$ and $\boldsymbol{b} = [b_1, ..., b_d]$, their **dot product** $\boldsymbol{a} \cdot \boldsymbol{b}$ is the real value $\sum_{i=1}^{d} a_i b_i$. This is sometimes also referred to as the *inner product* of $\boldsymbol{a}$ and $\boldsymbol{b}$. Next, we will prove an important but less trivial property of dot product:

Lemma 1. *If* $\boldsymbol{a} \neq \boldsymbol{0}$ *and* $\boldsymbol{b} \neq \boldsymbol{0}$*, then* $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \gamma$*, where* $\gamma \in [0°, 180°]$ *is the angle between non-zero vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*. .*

Proof. Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be the directed segments that define $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively; see Figure 1. We know that $\overrightarrow{AB}$ defines the vector $\boldsymbol{b} - \boldsymbol{a}$. By the law of cosine, we have:

$$\begin{aligned} |\overrightarrow{AB}|^2 &= |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}| \cos \gamma \quad \Rightarrow \\ \cos \gamma &= \frac{|\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - |\overrightarrow{AB}|^2}{2|\overrightarrow{OA}||\overrightarrow{OB}|} \qquad (1) \end{aligned}$$

[1] This is to say that the angle we want here never exceeds 180 degrees.

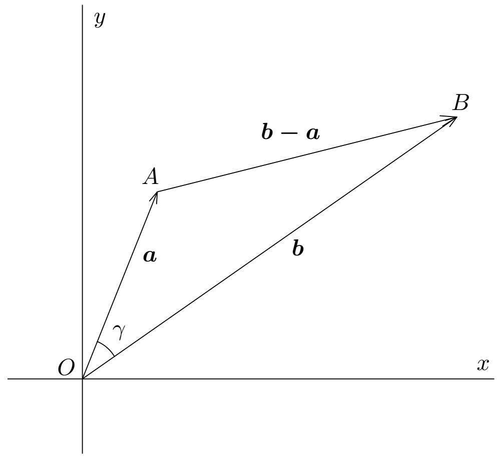


Figure 1: Proof of Lemma 1

On the other hand, we have:

$$\begin{aligned}
|\overrightarrow{OA}|^2 &= |\boldsymbol{a}|^2 = \boldsymbol{a}\cdot\boldsymbol{a} \\
|\overrightarrow{OB}|^2 &= |\boldsymbol{b}|^2 = \boldsymbol{b}\cdot\boldsymbol{b} \\
|\overrightarrow{AB}|^2 &= |\boldsymbol{b}-\boldsymbol{a}|^2 = (\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a}) \\
\text{(by distributivity of dot product)} &= (\boldsymbol{b}-\boldsymbol{a})\cdot\boldsymbol{b} - (\boldsymbol{b}-\boldsymbol{a})\cdot\boldsymbol{a} \\
\text{(by distributivity of dot product)} &= \boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{a} \\
&= \boldsymbol{b}\cdot\boldsymbol{b} - 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{a}
\end{aligned}$$

we can derive from (1)

$$\cos\gamma = \frac{\boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{b}\cdot\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{b} - 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{a})}{2|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

thus completing the proof. □

Corollary 1. *When $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b} \neq \boldsymbol{0}$, then $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal.*

Dot Product and Projection Length. Let us now see an important use of dot product: computing the projection length of a line segment. Figure 2 shows 3 points $P(-5, 7, 2)$, $A(3, 20, 8)$, and $B(1, 10, 5)$. Let C be the projection of point A onto $\overrightarrow{PB}$. We want to calculate the length of $\overrightarrow{PC}$, denoted as $|\overrightarrow{PC}|$.

Dot products provide an easy way to solve this problem. Let $\boldsymbol{a}$ be the vector defined by $\overrightarrow{PA}$, and $\vec{b}$ the vector defined by $\overrightarrow{PB}$. Clearly, $\boldsymbol{a} = [8, 13, 6]$ and $\boldsymbol{b} = [6, 3, 3]$. It thus follows that $\boldsymbol{a}\cdot\boldsymbol{b} = [8\cdot 6 + 13\cdot 3 + 6\cdot 3] = 105$. On the other hand, from Lemma 1, we know that $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\gamma$, where γ is the angle as shown in Figure 2b. As $|\boldsymbol{b}| = \sqrt{54}$, we know that

$$\begin{aligned}
|\boldsymbol{a}|\sqrt{54}\cos\gamma = 105 \quad &\Rightarrow \\
|\boldsymbol{a}|\cos\gamma = 105/\sqrt{54}&.
\end{aligned}$$

Observe from Figure 2b $|\boldsymbol{a}|\cos\gamma$ is exactly $|\overrightarrow{PC}|$.

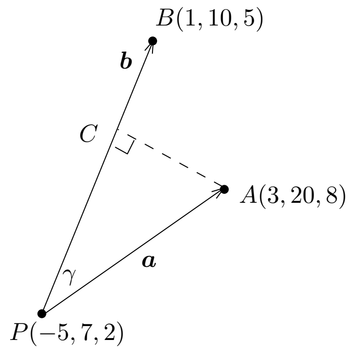


Figure 2: Using dot product to calculate projection lengths

3 Cross Product

Unlike dot product which is defined on vectors of arbitrary dimensionality d, cross product is defined only on 3d vectors:

Definition 2. *Given two 3d vectors $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$, we define $\boldsymbol{a} \times \boldsymbol{b}$, which is called the* **cross product** *of $\boldsymbol{a}$ and $\boldsymbol{b}$, as the vector $\boldsymbol{c} = [c_1, c_2, c_3]$ where*

$$\begin{aligned} c_1 &= a_2 b_3 - a_3 b_2 \\ c_2 &= a_3 b_1 - a_1 b_3 \\ c_3 &= a_1 b_2 - a_2 b_1. \end{aligned}$$

The following equation offers an easy way to remember the above equations:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

It is easy to verify by definition the following properties of cross product:

- (Anti-Commutativity) $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.
- (Distributivity) $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$, and $(\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a} = (\boldsymbol{b} \times \boldsymbol{a}) + (\boldsymbol{c} \times \boldsymbol{a})$.

Note that in general cross product does *not* necessarily obey associativity. Here is a counter example: $\boldsymbol{i} \times \boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{0} \times \boldsymbol{j} = \boldsymbol{0}$, but $\boldsymbol{i} \times (\boldsymbol{i} \times \boldsymbol{j}) = \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$.

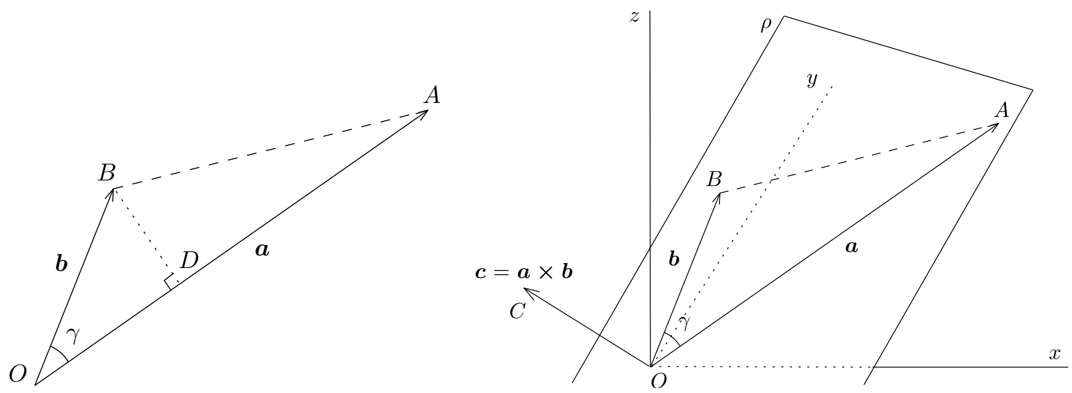
Geometry of Cross Products. Next we will gain a geometric understanding about cross products.

Lemma 2. *Let $\gamma \in [0°, 180°]$ be the angle between the directions of two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. Then, $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \gamma$.*

Proof. See appendix. □

As an immediate corollary, we know that $\boldsymbol{c} = \boldsymbol{0}$ in each of the following scenarios:

- $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$.



(a) Length of $\boldsymbol{a} \times \boldsymbol{b}$ (b) Direction of $\boldsymbol{a} \times \boldsymbol{b}$

Figure 3: Illustration of cross product

- The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $0°$ or $180°$.

If $\boldsymbol{c} \neq \mathbf{0}$, its length $|\boldsymbol{c}|$ has a beautiful explanation. Let O be the origin; and let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ the directed segments that define $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Then, $|\boldsymbol{c}|$ is twice the area of the triangle OAB; see Figure 3a (note that the length of directed segment $\overrightarrow{BD}$ equals $|\boldsymbol{b}| \sin \gamma$).

Lemma 3. *Let $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. Then, $\boldsymbol{a} \cdot \boldsymbol{c} = 0$ and $\boldsymbol{b} \cdot \boldsymbol{c} = 0$.*

Proof. Let $\boldsymbol{a} = [a_1, a_2, a_3]$, $\boldsymbol{b} = [b_1, b_2, b_3]$, and $\boldsymbol{c} = [c_1, c_2, c_3]$. We will prove only $\boldsymbol{a} \cdot \boldsymbol{c} = 0$ because an analogous argument shows $\boldsymbol{b} \cdot \boldsymbol{c} = 0$.

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{c} &= a_1c_1 + a_2c_2 + a_3c_3 \\ &= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) \\ &= 0. \end{aligned}$$

□

The lemma leads to the following important corollary:

Corollary 2. *Let $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. If $\boldsymbol{c} \neq \mathbf{0}$, then the directed segment $\overrightarrow{OC}$ defining $\boldsymbol{c}$ is perpendicular to the plane determined by the directed segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ that define $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively (see Figure 3b, where the plane is ρ).*

Proof. Since $\boldsymbol{c} \neq \mathbf{0}$, we know that (i) neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ is $\mathbf{0}$, and (ii) the angle γ between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ is larger than $0°$ but smaller than $180°$. Hence, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ uniquely determine a plane ρ. Since $\boldsymbol{a} \cdot \boldsymbol{c} = 0$ and $\boldsymbol{b} \cdot \boldsymbol{c} = 0$, we know that $\overrightarrow{OC}$ is orthogonal to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Hence, $\overrightarrow{OC}$ is perpendicular to ρ. □

We are almost ready to explain $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ in a way much more intuitive than Definition 2. Recall that to unambiguously pinpoint a vector, we need to specify (i) its length, and (ii) its direction. Lemma 2 has given the length, and Corollary 2 has *almost* given its direction. Why did we say "almost"? Because there are two directed segments emanating from the origin that are perpendicular to the plane ρ in Figure 3b: besides the $\boldsymbol{c}$ shown, $-\boldsymbol{c}$ is also perpendicular to ρ.

We can remove this last piece of ambiguity as follows. Let us see the plane ρ from the side such that $\boldsymbol{c}$ shoots into our eyes. The direction of $\boldsymbol{a}$ should turn *counter-clockwise* to the direction of $\boldsymbol{b}$ by an angle less than 180° (i.e., γ in Figure 3b). Notice that if we see the plane ρ from the wrong side, then $\boldsymbol{a}$ needs to do so *clockwise* to reach $\boldsymbol{b}$. At this point, we have obtained a complete geometric description about $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$.

Appendix

Proof of Lemma 2

Let $\boldsymbol{a} = [a_1, a_2, a_3]$, $\boldsymbol{b} = [b_1, b_2, b_3]$, and $\boldsymbol{c} = [c_1, c_2, c_3]$ (remember $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$). We will first establish another lemma which is interesting in its own right:

Lemma 4. $(|\boldsymbol{a}||\boldsymbol{b}|)^2 = |\boldsymbol{c}|^2 + (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

Proof. We will take a bruteforce approach to prove the lemma, by representing all the quantities in the target equation with coordinates.

$$\begin{aligned}
(|\boldsymbol{a}||\boldsymbol{b}|)^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&= a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2 \\
|\boldsymbol{a} \times \boldsymbol{b}|^2 &= c_1^2 + c_2^2 + c_3^2 \\
&= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 - 2a_2b_2a_3b_3 - 2a_1b_1a_3b_3 - 2a_1b_1a_2b_2 \\
(\boldsymbol{a} \cdot \boldsymbol{b})^2 &= (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3
\end{aligned}$$

The lemma thus follows. □

Now we proceed to prove Lemma 2. From Lemma 1, we know that $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\gamma$. Hence:

$$\begin{aligned}
(|\boldsymbol{a}||\boldsymbol{b}|)^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 &= (|\boldsymbol{a}||\boldsymbol{b}|)^2 - (|\boldsymbol{a}||\boldsymbol{b}|)^2 \cos^2\gamma \\
&= (|\boldsymbol{a}||\boldsymbol{b}|)^2 (1 - \cos^2\gamma) \\
&= (|\boldsymbol{a}||\boldsymbol{b}|)^2 \sin^2\gamma.
\end{aligned}$$

By combining the above with Lemma 4, we obtain:

$$|\boldsymbol{c}|^2 = (|\boldsymbol{a}||\boldsymbol{b}|)^2 \sin^2\gamma.$$

Since $\sin\gamma \geq 0$ (recall that $\gamma \in [0°, 180°]$), it follows that $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\gamma$.