Lecture Notes: Line Integrals by Arc Length

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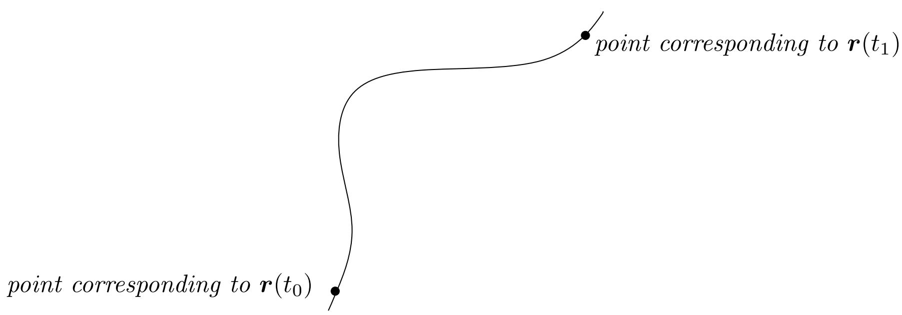
Consider a smooth curve in $\mathbb{R}^d$ given by the vector function $\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$, and the arc C from t_0 to t_1, an example of which is given below:



Let $f(x_1, x_2, ..., x_d)$ be a scalar function. Given point p with coordinates $(x_1, ..., x_d)$, we will use $f(p)$ as a short form of $f(x_1, x_2, ..., x_d)$. We now define a new form of integrals:

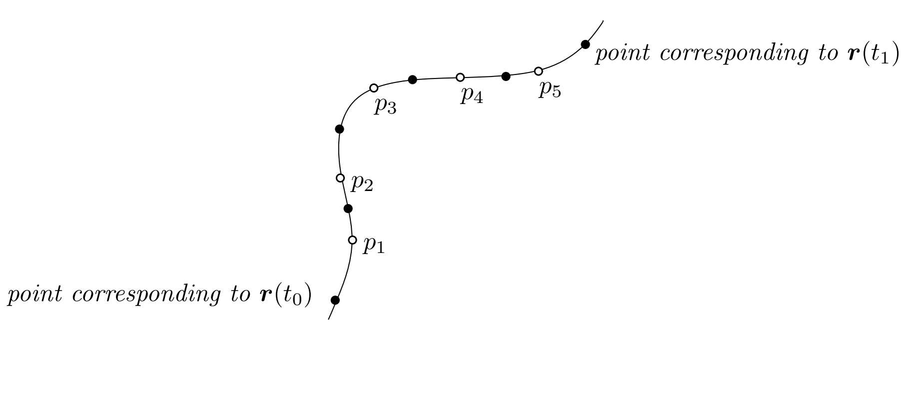
Definition 1. *Evenly divide the interval $[t_0, t_1]$ by inserting $n+1$ break points $\tau_0, \tau_1, \tau_2, ..., \tau_n$ where $\tau_0 = t_0$ and $\tau_i - \tau_{i-1} = (t_1 - t_0)/n$ for each $i \in [1, n]$. For each i, define Δs_i as the length of the arc from point $\boldsymbol{r}(\tau_{i-1})$ to $\boldsymbol{r}(\tau_i)$, and take an arbitrary point p_i on the arc. If the following limit exists:*

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(p_i) \cdot \Delta s_i$$

then we define

$$\int_C f(x_1, ..., x_d) ds \quad = \quad \lim_{n \to \infty} \sum_{i=1}^{n} f(p_i) \cdot \Delta s_i. \tag{1}$$

The integral in the left hand side of (1) is called *line integral by arc length*. The figure below illustrates the definition with $n = 5$.



As a special case, when $f(x_1, ..., x_d) = 1$, we have:

$$\int_C ds = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta s_i = \text{length of } C.$$

A line integral is almost always evaluated by changing the integral variable s to t, as demonstrated in the following examples.

Example 1. Consider the circle $x^2 + y^2 = 1$. Let C be the arc on the circle from $(1, 0)$ to $(-1, 0)$. Next we show how to calculate the line integral

$$\int_C x + y \; ds.$$

C can be represented as the set of $[x(t), y(t)]$ where

$$\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin t. \end{aligned}$$

and t ranges from 0 to π. Denote by L the length of C. It is worth pointing out that we will never need to find out the value of L, whose purpose is merely to indicate the range of s, as is clear in the derivation below:

$$\begin{aligned} \int_C x + y \; ds &= \int_0^L x + y \; ds \\ &= \int_0^\pi (x + y) \frac{ds}{dt} \; dt \\ &= \int_0^\pi (x + y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt \\ &= \int_0^\pi (\cos t + \sin t) \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt \\ &= \int_0^\pi (\cos t + \sin t) \, dt = 2. \end{aligned}$$

□

Example 2. Consider the helix $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ where

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \\ z(t) &= t. \end{aligned}$$

Let C be the curve from $t = 0$ to $t = \pi$. Next we show how to calculate

$$\int_C x + y + z \, ds.$$

Again, the main idea is to change s into t:

$$\begin{aligned}
\int_C x^2 + y + z\,ds &= \int_0^\pi (x(t) + y(t) + z(t))\,\frac{ds}{dt}dt \\
&= \int_0^\pi (x(t) + y(t) + z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\
&= \int_0^\pi (\cos(t) + \sin(t) + t)\sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}\,dt \\
&= \sqrt{2}\int_0^\pi \cos(t) + \sin(t) + t\,dt \\
&= \sqrt{2}(2 + \pi^2/2).
\end{aligned}$$

□