Lecture Notes: Green's Theorem

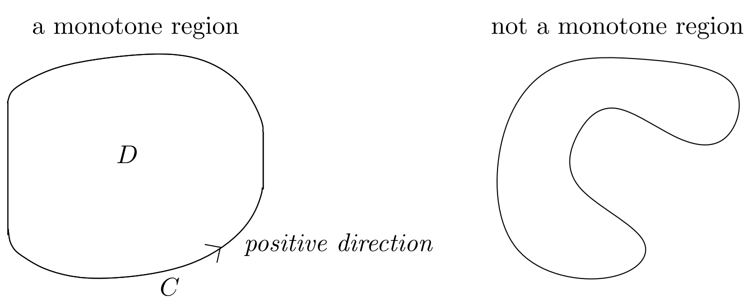
Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

In general, a curve C has a starting point p and an ending point q. However, it is possible that $p = q$, i.e., the starting point coincides with the ending point, in which case C is a *closed curve*. In this lecture, we will see a beautiful relationship between 2D line integrals on closed curves and double integrals.

1 Monotone Regions

Let C be a piecewise-smooth closed curve in $\mathbb{R}^2$, and D be the region that is enclosed by C. We say that D is *monotone* if it satisfies both of the following conditions:

- any vertical line intersects C into two points, unless the line passes the leftmost or rightmost point of C;
- any horizontal line intersects C into two points, unless the line passes the top-most or bottom-most point of C.



Suppose that D is monotone. We designate the *positive direction* of C as the counterclockwise direction. Choose an arbitrary point p on C, and denote the same point p also as q. We can view C instead as a curve obtained by walking from p counterclockwise along the boundary of D until hitting q.

We will now prove the first version of the Green's Theorem:

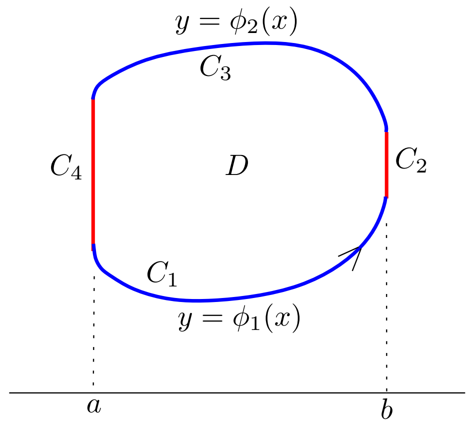
Theorem 1 (Green's Theorem). *Let $f_1(x, y)$ and $f_2(x, y)$ be scalar functions such that $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ are continuous in D. Then:*

$$\int_C f_1\,dx + f_2\,dy \quad = \quad \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\,dxdy. \tag{1}$$

Proof. We will first prove that

$$\int_C f_1\,dx \quad = \quad -\iint_D \frac{\partial f_1}{\partial y}\,dxdy. \tag{2}$$

Let a (and b) be the minimum (and maximum, resp.) x-coordinate of the points on C. Any monotone D can be regarded as the region between two curves: $y = \phi_1(x)$ and $y = \phi_2(x)$, for the range $x \in [a, b]$. Without loss of generality, let $y = \phi_1(x)$ be the lower curve, and $y = \phi_2(x)$ the upper curve, as shown as the blue curves below:



We break C into a sequence of C_1, C_2, C_3 and C_4. Note that C_2 and C_4 are vertical segments (shown above in red). Therefore:

$$\begin{aligned}
\int_C f_1\,dx &= \int_{C_1} f_1\,dx + \int_{C_2} f_1\,dx + \int_{C_3} f_1\,dx + \int_{C_4} f_1\,dx \\
&= \int_{C_1} f_1\,dx + \int_{C_3} f_1\,dx \\
&= \int_a^b f_1(x, \phi_1(x))\,dx + \int_b^a f_1(x, \phi_2(x))\,dx \\
&= \int_a^b f_1(x, \phi_1(x)) - f_1(x, \phi_2(x))\,dx.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\iint_D \frac{\partial f_1}{\partial y}\,dxdy &= \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f_1}{\partial y}\,dy \right) dx \\
&= \int_a^b f_1(x, \phi_2(x)) - f_1(x, \phi_1(x))\,dx. \\
&= -\int_C f_1\,dx
\end{aligned}$$

which establishes (2).

By repeating the above argument with respect to the y-dimension, we get

$$\int_C f_2\,dy = \iint_D \frac{\partial f_2}{\partial x}\,dxdy. \qquad (3)$$

Putting together (2) and (3) proves (1). □

As a special case, setting $f_1(x, y) = -y$ and $f_2(x, y) = x$, we obtain from (1):

$$\int_C (-y\,dx + x\,dy) = 2\iint_D dxdy. \qquad (4)$$

Note that the right hand side of the above is twice the area of D.

Example 1. Calculate the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. Let C be the ellipse's boundary, and D the ellipse itself. We know from (4) that

$$area(D) = \frac{1}{2} \int_C (-y\, dx + x\, dy).$$

Introduce $x(t) = a \cos t$ and $y(t) = b \sin t$. We have from the above that

$$\begin{aligned} area(D) &= \frac{1}{2} \int_0^{2\pi} -b \sin(t) \frac{dx}{dt} + a \cos(t) \frac{dy}{dt}\, dt. \\ &= \frac{1}{2} \int_0^{2\pi} ab \sin^2(t) + ab \cos^2(t)\, dt. \\ &= ab\pi. \end{aligned}$$

It may be interesting for you to evaluate $\iint_D dxdy$ directly without converting it to a line integral, and compare the amount calculation of the two solutions. □

Example 2. Let D be the square $[-1, 1] \times [-1, 1]$ (namely, x-projection $[-1, 1]$ and y-projection $[-1, 1]$). Let C be the boundary of D in the positive direction. Calculate $\int_C 6y^2\, dx + 2x - 2y^4\, dy$.

Solution. Let $f_1(x, y) = 6y^2$ and $f_2(x, y) = 2x - 2y^4$. By Theorem 1, we have:

$$\begin{aligned} \int_C (6y^2\, dx + 2x - 2y^4\, dy) &= \iint_D 2 - 12y\, dxdy \\ &= \iint_D 2\, dxdy - \iint_D 12y\, dxdy \\ &= 8 - \int_{-1}^{1} \left(12y \int_{-1}^{1} dx \right) dy \\ &= 8 - \int_{-1}^{1} 24y\, dy = 8. \end{aligned}$$

□

Remark. Notice from the above examples that in a line integral with a closed curve C we do not specify where C starts and ends explicitly. The reason is clear from Theorem 1: it does not matter! You can break C at any point p, and treat it as a curve that starts from p, goes a round, and then ends at p. The line integral is always the same regardless of your choice.

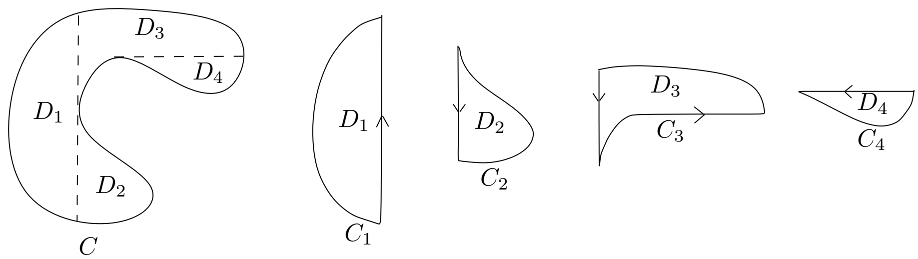
2 Green's Theorem for Non-Monotone Regions

Next, we extend Theorem 1 to any closed region D whose boundary is a piecewise-smooth curve.

Regions without Holes. Let D be a (possibly non-monotone) region enclosed by a closed piecewise-smooth curve C. As before, we designate the *positive direction* of C as the counter-clockwise direction.

Theorem 2. *Theorem 1 still holds even if C is not monotone.*

We will not prove the theorem formally, but we can gain the key idea from the example below. The leftmost figure is a non-monotone region D enclosed by curve C. Let us break it with two dashed line segments into 4 regions D_1, D_2, D_3, and D_4, each of which is monotone.



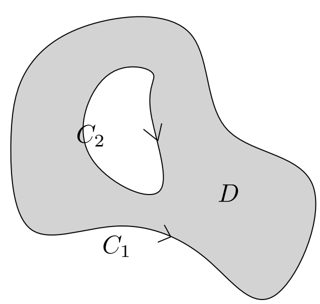
Let $C_1, C_2, ..., C_4$ be the boundary curves of $D_1, D_2, ..., D_4$, respectively. Applying Theorem 1 on each curve, we get:

$$\sum_{i=1}^{4} \int_{C_i} (f_1\, dx + f_2\, dy) = \sum_{i=1}^{4} \iint_{D_i} \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy.$$

$$\Rightarrow \sum_{i=1}^{4} \int_{C_i} (f_1\, dx + f_2\, dy) = \iint_{D} \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy.$$

Interestingly, the left hand side equals $\int_C (f_1\, dx + f_2\, dy)$! Notice that every dashed line is integrated exactly twice with *opposite* directions!

Regions with Holes. Now consider D to be any *connected region*, i.e., namely, we can move from a point in D to any other point in D without leaving D. Note that D may contain "holes"; for example, see the figure below. We define the *boundary* of D as the set of points p in D such that, *any* circle centered at p with an arbitrarily small radius must contain some points not belonging to D. In the figure below, the boundary of D consists of two curves C_1 and C_2.

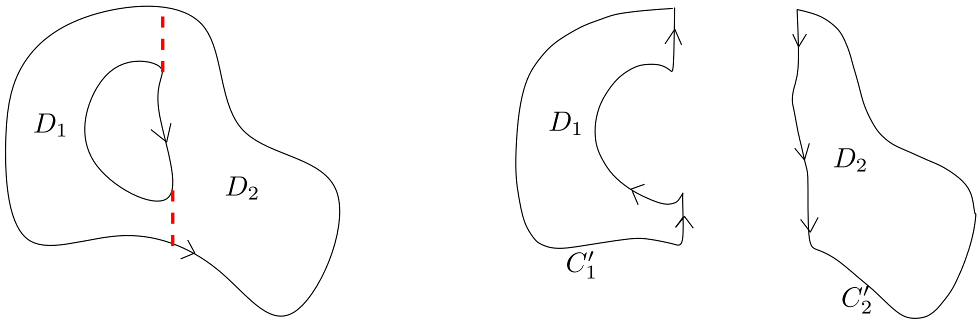


Consider, in general, that the boundary C of D is a set of closed piecewise smooth curves $C_1, C_2, ..., C_k$ for some finite value k (e.g., $k = 2$ in the above figure). For each C_i $(1 \le i \le k)$, we define its *positive direction* as follows: if we walk along that direction, then D is on our left hand side at all times. In the above example, the positive direction of C_1 is the counterclockwise direction, while that of C_2 is the clockwise direction.

We now present the Green's theorem in its most general form:

Theorem 3. *Theorem 1 still holds on the connected region D and its boundary C defined as above.*

Again, we omit a formal proof of the theorem, but illustrate the key idea using an example. Consider the region D demonstrated earlier. We can cut it into two regions, neither of which has a hole as shown below:

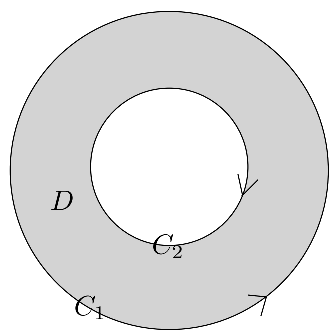


Let C_1', C_2' be the boundaries of D_1 and D_2, respectively. We know

$$\begin{aligned}\sum_{i=1}^{2}\int_{C_i'}(f_1\,dx+f_2\,dy) &= \sum_{i=1}^{2}\iint_{D_i}\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\,dxdy.\\ \Rightarrow \sum_{i=1}^{2}\int_{C_i'}(f_1\,dx+f_2\,dy) &= \iint_{D}\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\,dxdy.\end{aligned}$$

The left hand side equals $\int_C(f_1\,dx+f_2\,dy)$, noticing that every dashed line is integrated exactly twice with *opposite* directions.

Example 3. Let C_1 be the circle $x^2+y^2=10$, and C_2 be the circle $x^2+y^2=5$. Let D be the region between the two circles (i.e., the shaded area in the figure below). Let $C=\{C_1,C_2\}$ be the boundary of D with C_1, C_2 in the positive direction.



It is clear that $area(D)=10\pi-5\pi=5\pi$. Next, we will calculate the $area(D)$ by line integral. According to Theorem 3, we have:

$$\begin{aligned}area(D)=\iint_D dxdy &= \frac{1}{2}\int_C(-y\,dx+x\,dy)\\ &= \frac{1}{2}\left(\int_{C_1}(-y\,dx+x\,dy)+\int_{C_2}(-y\,dx+x\,dy)\right). \qquad (5)\end{aligned}$$

Represent C_1 in the parametric form $[\sqrt{10}\cos(u), \sqrt{10}\sin(u)]$. Then:

$$\begin{aligned}
\int_{C_1} (-y\,dx + x\,dy) &= \int_0^{2\pi} -\sqrt{10}\sin(u)\frac{dx}{du} + \sqrt{10}\cos(u)\frac{dy}{du}\,du \\
&= \int_0^{2\pi} (-\sqrt{10}\sin(u))^2 + (\sqrt{10}\cos(u))^2\,du \\
&= 20\pi.
\end{aligned}$$

Represent C_2 in the parametric form $[\sqrt{5}\cos(v), \sqrt{5}\sin(v)]$. Then:

$$\begin{aligned}
\int_{C_2} (-y\,dx + x\,dy) &= \int_{2\pi}^{0} -\sqrt{5}\sin(v)\frac{dx}{dv} + \sqrt{5}\cos(v)\frac{dy}{dv}\,dv \\
&= \int_{2\pi}^{0} (-\sqrt{5}\sin(v))^2 + (\sqrt{5}\cos(v))^2\,dv \\
&= -10\pi.
\end{aligned}$$

Therefore, (5) evaluates to $\frac{1}{2}(20\pi - 10\pi) = 5\pi$. □