Exercises: Dimensions, Spans, and Linear Transformations

In the following exercises, $\mathbb{R}$ denotes the set of all real numbers.

Problem 1. Let V be the set of following 1×4 vectors:

$$[3, 0, 1, 2]$$
$$[6, 1, 0, 0]$$
$$[12, 1, 2, 4]$$
$$[6, 0, 2, 4]$$
$$[9, 0, 1, 2]$$

Find the dimension of V.

Problem 2. Let V be the set of 1×4 vectors $[2x - 3y, x + 2y, -y, 4x]$ with $x, y \in \mathbb{R}$. Find the dimension of V and give a basis of V.

Problem 3. For each set V of vectors given below, find its dimension and give a basis:

- (a) V is the set of 2D points given by $y = x$ (here, we regard each point (x, y) as a 1×2 vector $[x, y]$);
- (b) V is the set of 2D points given by $y = x + 1$.

Problem 4. Let V_1 be the set of vectors $[x_1, x_2]^T$ where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. Define:

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 \\ y_2 &= 4x_1 + x_2 \end{aligned}$$

Let V_2 be the set of vectors $[y_1, y_2]^T$ obtained by applying the above to all vectors $[x_1, x_2]^T \in V_1$. Answer the following questions:

(a) Give the matrix $\boldsymbol{A}$ in the linear transformation $[y_1, y_2]^T = \boldsymbol{A}[x_1, x_2]^T$ from V_1 to V_2.

(b) It is known that there is a linear transformation $[x_1, x_2]^T = \boldsymbol{A'}[y_1, y_2]^T$ from V_2 to V_1. Give the details of the matrix $\boldsymbol{A'}$.

Problem 5. Let V be a set of $1 \times n$ vectors. Let V' be the *projection* of V on the first $t < n$ components, namely:

$$V' = \Big\{[x_1, x_2, ..., x_t] \;\Big|\; [x_1, x_2, ..., x_t, x_{t+1}, ..., x_n] \in V\Big\}.$$

Prove: the dimension of V is at least the dimension of V'.

For example, if V is the set of 5 vectors in Problem 1 and $t = 2$, then V' is the set of following vectors:

$$[3, 0]$$
$$[6, 1]$$
$$[12, 1]$$
$$[6, 0]$$
$$[9, 0].$$

Problem 6 (Hard). Consider the following system of linear equations:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let V be the set of 5×1 vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ that satisfy the equation. Prove that V has dimension 2, and find a basis of V.

Problem 7 (Hard). Consider the following linear system about $\boldsymbol{x}$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$$

where $\boldsymbol{A}$ is an $m \times n$ coefficient matrix, and $\boldsymbol{x}$ an $n \times 1$ matrix. Let V be the set of all such $\boldsymbol{x}$ satisfying the system. Suppose that the rank of $\boldsymbol{A}$ is $r < n$. Prove that V has dimension $n - r$.