Grid Decomposition

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This lecture will introduce grid decomposition, which is a fundamental technique for solving many computational geometry problems. We will demonstrate the technique by using it to solve the closest pair and close pairs problems.

Closest Pair and Close Pairs

Let P be a set of points $\mathbb{R}^d$. The objective of the **closest pair problem** is to output a pair of distinct points $p, q \in P$ that have the smallest distance to each other, or formally:

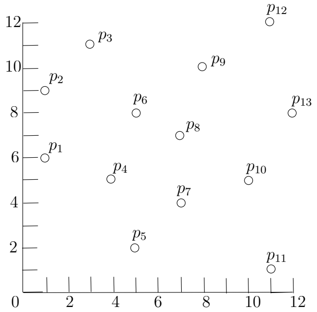
$$dist(p, q) \quad = \quad \min_{p_1, \, p_2 \, \in \, P, \;\; p_1 \, \neq \, p_2} dist(p, q).$$

where $dist(.,.)$ represents the Euclidean distance of two points.

Let P be a set of points $\mathbb{R}^d$, and r a real value. The objective of the **close pairs problem** is to output all pairs of distinct points $p, q \in P$ satisfying:

$$dist(p, q) \quad \leq \quad r.$$

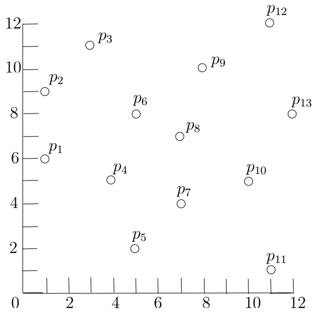
Example: Closest Pair



The answer is (p_6, p_8).

Example: Close Pairs

Assume $r = 4\sqrt{2}$.

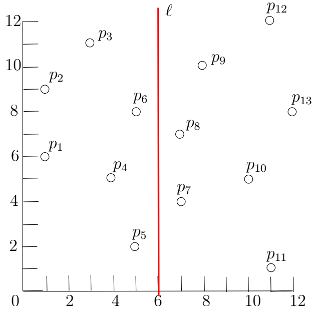


The answer is $\{(p_1, p_4), (p_1, p_2), (p_2, p_3), (p_2, p_6), (p_2, p_4), ...\}$.

We will discuss first the closest pair problem and then the close pairs problem. Note that both problems can be easily solved in $O(n^2)$ time where $n = |P|$. We will settle the first problem in $O(n \log n)$ expected time and the second in $O(n + k)$ expected time, where k is the number of pairs reported.

Closest Pair in 2D

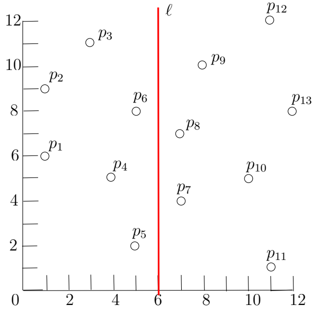
Let us now turn our attention to 2D. Naturally, we divide P evenly using a vertical line ℓ, such that there are $n/2$ points on each side. Let P_1 (or P_2) be the set of points on the left (or right) of ℓ. We recursively find the closest pair in P_1, and then in P_2, respectively.



In the above example, the closest pair of P_1 is (p_2, p_3), and that of P_2 is (p_7, p_8).

Closest Pair in 2D

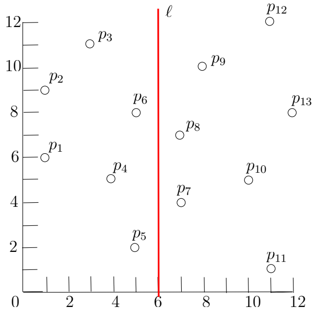
We need to "merge" the two halves to find the global closest pair. It suffices to find the closest pair (p_1, p_2) satisfying $p_1 \in P_1$ and $p_2 \in P_2$ — namely, p_1, p_2 come from different sides. Call it the **crossing** closest pair.



In the above example, the crossing closest pair is (p_6, p_8). The global closest pair must be among the two "local" pairs (p_2, p_3), (p_7, p_8), and the crossing pair (p_6, p_8).

Closest Pair in 2D

We now explain how to find the crossing closest pair. Let r_1 be the distance of the closest pair in P_1, and r_2 be the distance of the closest pair in P_2. Define $r = \min\{r_1, r_2\}$.

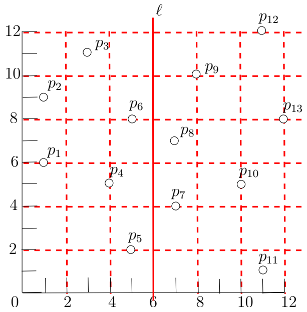


In the above example, $r_1 = \sqrt{8}$, $r_2 = 3$, and $r = \min\{r_1, r_2\} = \sqrt{8}$.

Observation: We care about the crossing closest pair only if its distance is smaller than or equal to r.

Closest Pair in 2D

Impose an arbitrary grid G onto the data space, where (i) each cell is an axis-parallel square with side length $r/\sqrt{2}$, and (ii) ℓ is a line in the grid.



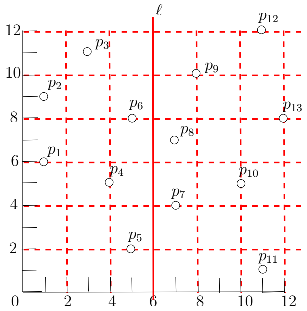
Each point p can be covered by at most 4 cells (e.g., p_9 is covered by 4 cells, but p_8 by only 1 cell).

Closest Pair in 2D

For each cell c, we denote by $c(P)$ the set of points in P that are covered by c.

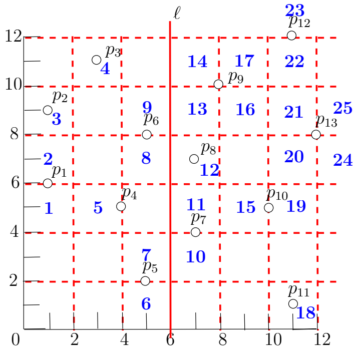
Observation: For every c, $|c(P)| \leq 2 = O(1)$!

Proof: Note that the diagonal of c has length r. If c covers more than 2 points, at least 2 points have distance less than r, contradicting the definition of r. □



Closest Pair in 2D

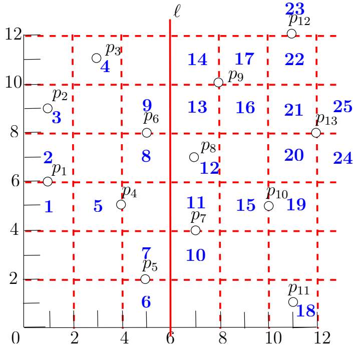
Group the points by the cells they belong. A cell is **non-empty** if it covers at least one point. There can be at most $4n$ non-empty cells.



In the above example, there are 25 non-empty cells.

Closest Pair in 2D

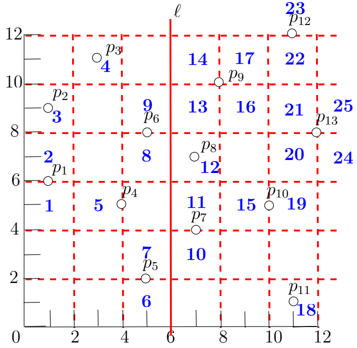
Each cell can be uniquely identified by the coordinates of its centroid, which we refer to as the **id** of the cell. Using hashing, by spending $O(n)$ expected time in total, we can create for each cell c, a linked list containing all the points in $c(P)$ (i.e., the set of points covered by c).



Closest Pair in 2D

Let c_1, c_2 be two non-empty cells. We say that c_1 is an bluer-neighbor of c_2 (and vice versa) if their mindist is at most r.

To find a crossing closest pair within distance r, it suffices to consider (the points in) non-empty cells c_1, c_2 satisfying (i) c_1 is on the left of ℓ, and c_2 is on the right, and (ii) c_1 and c_2 are r-neighbors.

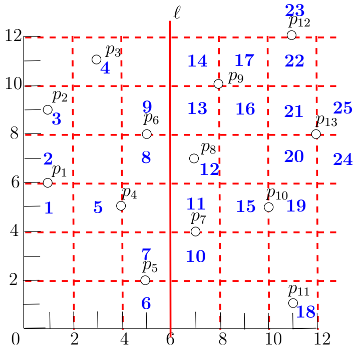


For example, Cell 11 is an r-neighbor of Cell 5, while Cell 15 is not. In other words, we need to consider the cell pair (5, 11), but not (5, 15).

Cells 1, 2, 3, 18, 19, 20, 21, 22, 23, 24, and 25 can be immediately discarded (why?).

Closest Pair in 2D

Observation: Each non-empty cell c on the left of ℓ has $O(1)$ r-neighbor cells on the right of ℓ.



For example, for Cell 8, we need to consider 8 pairs: (8, 10), (8, 11), (8, 12), (8, 13), (8, 14), (8, 15), (8, 16), (8, 17).

Closest Pair in 2D

The above discussion motivates the following algorithm for finding a crossing closest pair within distance r:

```
1. for every non-empty cell $c_1$ on the left of $\ell$
2.     for every $r$-neighbor cell $c_2$ of $c_1$ on the right of $\ell$
3.         calculate the distance of each pair of
           points $(p_1, p_2) \in c_1(P) \times c_2(P)$
4. return the closest one among all the pairs inspected at Line 3, if the
   pair has distance at most $r$.
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As mentioned, for each c_1, there are $O(1)$ cells c_2 that need to be considered. Since $c_1(P)$ and $c_2(P)$ each contain at most 2 points, each execution of Line 3 takes only $O(1)$ time. The overall algorithm takes $O(n)$ expected time in total.

Think: How to find the cells c_2 for each c_1 in $O(1)$ expected time? Hint: by hashing on the cell ids.

Closest Pair in 2D: Analysis

Let $f(n)$ be the expected running time of our algorithm, it follows that

$$f(n) \quad \leq \quad 2 \cdot f(n/2) + O(n)$$

while $f(n) = O(1)$ for $n \leq 2$.

The recurrence solves to $f(n) = O(n \log n)$.

The success behind our grid decomposition technique in the closest-pair problem comes from the property that, each cell in the grid has $O(1)$ r-neighbor cells.

We now proceed to tackle the close-pairs problem by essentially using the same property. Recall that our objective is to achieve $O(n+k)$ expected time, where k is the number of pairs reported.

Recall the definition of the close-pairs problem.

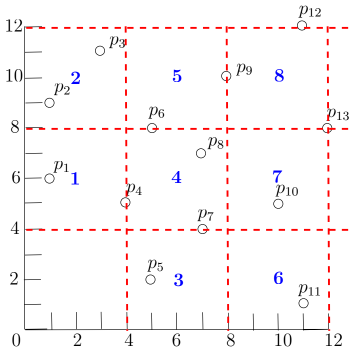
Let P be a set of distinct points $\mathbb{R}^d$, and r a real value. The objective of the close pairs problem is to output all pairs of distinct points $p, q \in P$ satisfying:

$$dist(p, q) \quad \leq \quad r.$$

We will focus on 2D space, because the algorithm can be directly extended to arbitrary dimensionalities.

Close Pairs in 2D

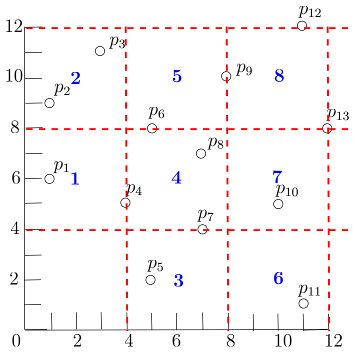
We will explain the algorithm using the same dataset and $r = 4\sqrt{2}$.



Step 1: Impose an arbitrary grid where each square cell has side length $r/\sqrt{2} = 4$. Identify all the non-empty cells (there are 8 such cells in our example).

Close Pairs in 2D

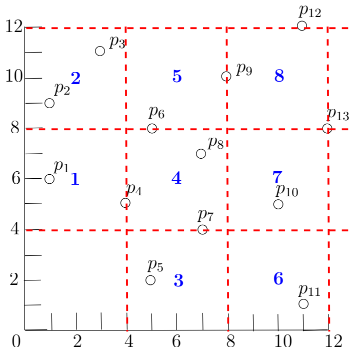
Step 2: For each cell c, let $c(P)$ be the set of points covered by c. Simply report all pairs of distinct points in $c(P)$ — notice that any two points in the same cell must have distance at most r.



For example, 1 pair is reported for Cell 1, and 3 pairs for Cell 8.

Close Pairs in 2D

Step 3: For each cell c_1, identify all of its r-neighbor cells c_2. For every c_2, inspect all pairs of distinct points $(p_1, p_2) \in c_1(P) \times c_2(P)$, and report the ones within distance at most r.



For example, from Cells 2 and 4, inspect all the 8 pairs in $\{p_2, p_3\} \times \{p_4, p_6, p_7, p_8\}$, and report $(p_2, p_4), (p_2, p_6), (p_3, p_6)$.

Close Pairs in 2D: Analysis

Next, we will prove that our algorithm runs in $O(n+k)$ expected time. At first glance, this may look a bit surprising. Recall that in Step 3, for each pair of r-neighbor cells (c_1, c_2), we spend a quadratic amount of time $O(|c_1(P)||c_2(P)|)$, but risk finding no answer pairs at all. Indeed, the core of the analysis is to show that the total time of doing so is bounded by $O(n+k)$.

We will focus on Steps 2 and 3 because Step 1 obviously takes $O(n)$ expected time (by hashing).

Close Pairs in 2D: Analysis (Step 2)

Let $c_1, c_2, ..., c_m$ be the non-empty cells, for some $m \geq 1$. Define $n_i = |c_i(P)|$, namely, the number of points covered by c_i, for each $i \in [1, m]$. Clearly $\sum_{i=1}^{m} n_i \geq n$.

The cost of Step 2 is obviously

$$\sum_{i=1}^{m} O(n_i^2)$$

Notice that

$$k \geq \sum_{i=1}^{m} n_i(n_i - 1)/2 = \frac{1}{2}\sum_{i=1}^{m} n_i^2 - \frac{1}{2}\sum_{i=1}^{m} n_i \geq \frac{1}{2}\left(\sum_{i=1}^{m} n_i^2\right) - \frac{n}{2}.$$

We thus have

$$\sum_{i=1}^{m} O(n_i^2) = O(n + k).$$

Close Pairs in 2D: Analysis (Step 3)

We will prove that the cost of Step 3 is $\sum_{i=1}^{m} O(n_i^2)$, and therefore, bounded by $O(n+k)$.

Let c_i and c_j be a pair of r-neighbor cells. Step 3 spends $O(n_i \cdot n_j)$ time to process $c_i(P) \times c_j(P)$. Clearly:

$$n_i \cdot n_j \leq (n_i^2 + n_j^2)/2.$$

Close Pairs in 2D: Analysis (Step 3)

The total cost of Step 3 can be written as

$$\begin{aligned} & O\left(\sum_{i=1}^{m} \sum_{j:\ c_j \text{ is an } r\text{-neighbor of } c_i} n_i \cdot n_j\right) \\ = \ & O\left(\sum_{i=1}^{m} \sum_{j:\ c_j \text{ is an } r\text{-neighbor of } c_i} (n_i^2 + n_j^2)\right) \\ & \text{(by the inequality of the previous slide)} \end{aligned}$$

As a cell has $O(1)$ r-neighbor cells, we know that each n_i^2 can appear only $O(1)$ times in the above summation. Therefore, the summation is bounded by $O(\sum_{i=1}^{m} n_i^2)$.

We now conclude that the running time of our close-pairs algorithm is $O(n + k)$ expected.