Edit Distances: Verification

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October 23, 2019

Given two strings s, t, we already know how to compute their edit distance $edit(s, t)$ using dynamic programming in $O(|s||t|)$ time. It turns out that we can do better if we only need to verify whether $edit(s, t) \leq d$. This can be done in

$$O(|s| + |t| + d \cdot \min\{|s|, |t|\})$$

time.

We will consider only $|s| = |t| = \ell$. The case of $|s| \neq |t|$ is similar and left to you.

Our goal now is to verify whether $edit(s, t) \leq d$ in $O(d\ell)$ time for $d < \ell$ (if $d \geq \ell$, the answer is trivially yes).

Recall that, in order to compute $edit(s, t)$ in $O(\ell^2)$ time, our strategy was to fill in an $(\ell + 1) \times (\ell + 1)$ array A. To solve the verification problem, we will adopt a similar strategy, except that we will fill in only a hexagon part of A, as explained next.

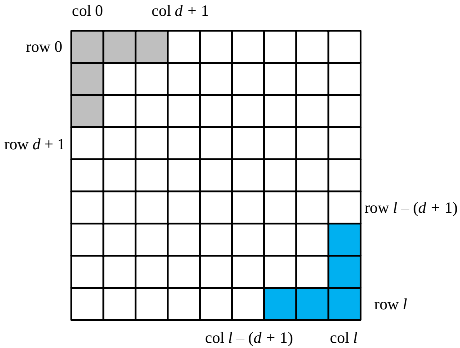
Let us first define the gray boundary cells to be

- At row 0, the left most $d + 1$ cells.
- At column 0, the top most $d + 1$ cells.

Define the blue boundary cells to be

- At row ℓ, the right most $d + 1$ cells.
- At column ℓ, the bottom most $d + 1$ cells.

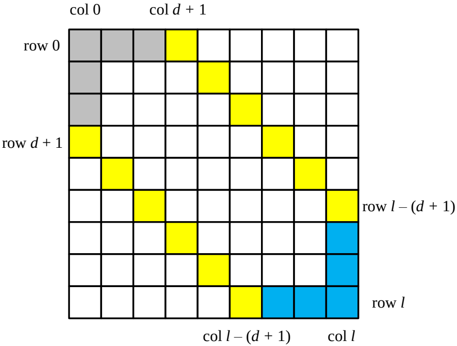
An example with $\ell = 8$ and $d = 2$:



Define the yellow boundary cells to be:

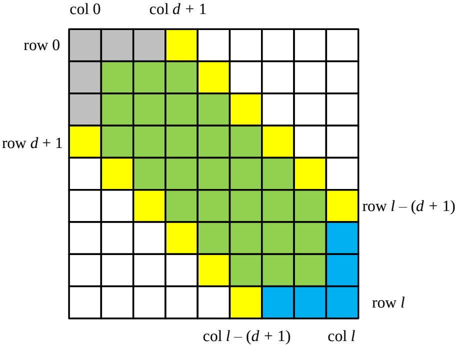
- $A[0, d+1]$, $A[1, d+2]$, ..., $A[\ell-(d+1), \ell]$
- $A[d+1, 0]$, $A[d+2, 1]$, ..., $A[\ell, \ell-(d+1)]$

An example with $\ell = 8$ and $d = 2$:



Define the green cells to be all those cells inside the region surrounded by the gray yellow, and blue boundary cells.

An example with $\ell = 8$ and $d = 2$:



We fill in only the colored cells (i.e., ignoring the others) as follows:

1. Fill in the gray cells normally.
2. Put $\geq d + 1$ in all the yellow cells.
3. Compute the green and blue cells in the same manner as in the $O(\ell^2)$-time algorithm (i.e., row by row, and left to right at each row).

Report yes if $A[\ell, \ell] \leq d$, and no, otherwise.

Since there are only $O(d\ell)$ colored cells, the running time is $O(d\ell)$.

Example: $s =$ `humanity`, $t =$ `hunamity`, and $d = 2$.

After the first two steps:

		h	u	m	a	n	i	t	y
	0	1	2	3					
h	1				$\geqslant 3$				
u	2					$\geqslant 3$			
n	3						$\geqslant 3$		
a		$\geqslant 3$						$\geqslant 3$	
m			$\geqslant 3$						$\geqslant 3$
i				$\geqslant 3$					
t					$\geqslant 3$				
y						$\geqslant 3$			

Edit distance by recurrence.

- If $m > 0$, $n > 0$, and $s[m] = t[n]$, then $edit(s, t)$ is:

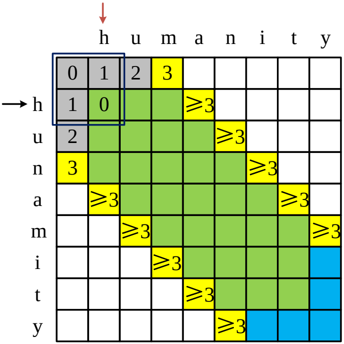
$$\min \begin{cases} 1 + edit(s, t[1..n-1]) \\ 1 + edit(s[1..m-1], t) \\ edit(s[1..m-1], t[1..n-1]) \end{cases} \tag{1}$$

- If $m > 0$, $n > 0$, and $s[m] \neq t[n]$, then $edit(s, t)$ is:

$$\min \begin{cases} 1 + edit(s, t[1..n-1]) \\ 1 + edit(s[1..m-1], t) \\ 1 + edit(s[1..m-1], t[1..n-1]) \end{cases} \tag{2}$$

Example: $s = \texttt{humanity}$, $t = \texttt{hunamity}$, and $d = 2$.

One more step:



Example: $s = \texttt{humanity}$, $t = \texttt{hunamity}$, and $d = 2$.

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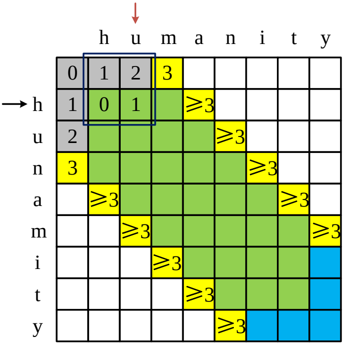

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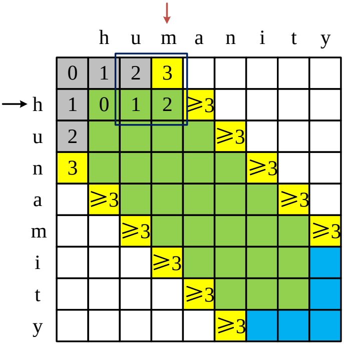

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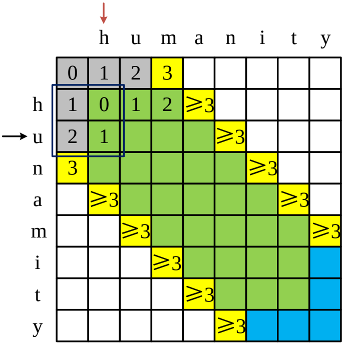


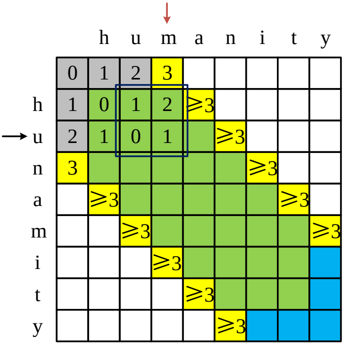
Example: $s = \texttt{humanity}$, $t = \texttt{hunamity}$, and $d = 2$.

One more step:

		h	u	m	a	n	i	t	y
	0	1	2	3					
h	1	0	1	2	$\geqslant 3$				
u	2	1	0			$\geqslant 3$			
n	3						$\geqslant 3$		
a		$\geqslant 3$						$\geqslant 3$	
m			$\geqslant 3$						$\geqslant 3$
i				$\geqslant 3$					
t					$\geqslant 3$				
y						$\geqslant 3$			

Example: $s = \texttt{humanity}$, $t = \texttt{hunamity}$, and $d = 2$.

One more step:



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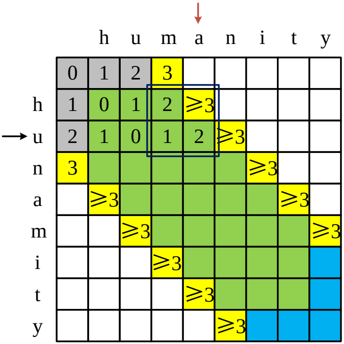

Example: $s = \texttt{humanity}$, $t = \texttt{hunamity}$, and $d = 2$.

After all steps:

		h	u	m	a	n	i	t	y
	0	1	2	3					
h	1	0	1	2	$\geqslant 3$				
u	2	1	0	1	2	$\geqslant 3$			
n	3	2	1	1	2	2	$\geqslant 3$		
a		$\geqslant 3$	2	2	1	2	3	$\geqslant 3$	
m			$\geqslant 3$	2	2	2	3	4	$\geqslant 3$
i				$\geqslant 3$	3	3	2	3	4
t					$\geqslant 3$	4	3	2	3
y						$\geqslant 3$	4	3	2

So we conclude $edit(s, t) \leq 2$.

Think

Why is the algorithm correct?