Computational Complexity 4: NP-Completeness of the Clique Decision Problem

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Recall:

The **NP-complete class** — denoted as NPC — is the set of decision problems π such that

- π is in NP;
- if π can be **solved** in polynomial time, then **every** problem in NP can be **solved** in polynomial time.

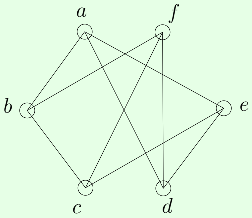
Once the first NPC problem has been found, it is much easier to prove the second.

Theorem: Let π^* be a decision problem in NPC. If π^* can be reduced to another decision problem π in polynomial time, then π must be NP-hard.
Hence, if π is also in NP, π is NP-complete.

Recall

The Clique Decision Problem: Let $G = (V, E)$ be an undirected graph. Given an integer k, decide whether we can find a set S of at least k vertices in V that are mutually connected (i.e., there is an edge between any two vertices in S).
Those k vertices and the edges among them form a **k-clique**.

Example: Consider



The answer is "yes" for $k \leq 3$, but "no" for $k \geq 4$.

Recall:

3-SAT

Variable: a boolean unknown x that can be assigned 0 or 1.
Literal: a variable x or its negation $\bar{x}$.
Clause: the OR of up to 3 literals.
Formula: the AND of clauses

The 3-SAT problem: Is there an assignment to the variables under which the formula evaluates to 1?
Such an assignment is called a **truth assignment**.

Example:

$$(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\bar{x_1} \vee \bar{x_4})$$

The answer is "yes". A certificate: $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$.

$$(x_1) \wedge (\bar{x_1} \vee x_2) \wedge (\bar{x_2})$$

The answer is "no".

Recall

Theorem: 3-SAT is NP-complete.

Next we will prove the NP-completeness of the clique decision problem with a reduction **from** the 3-SAT problem. Specifically, we will prove:

Theorem: If we have an algorithm $\mathcal{A}$ solving the clique decision problem in polynomial time, we can solve the 3-SAT problem using $\mathcal{A}$ in polynomial time.

The next few slides serve as a proof of the theorem.

Given an input to 3-SAT — namely a formula F with k clauses — we will construct a graph $G(V, E)$ such that F has a truth assignment **if and only if** G has a k-clique.

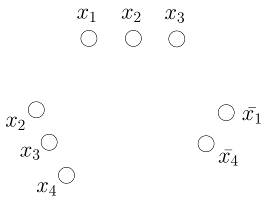
We construct $G(V, E)$ as follows:

- For each clause, create a vertex in V for every literal in the clause.
- For each pair of distinct vertices $u, v \in V$, create an edge $\{u, v\}$ in E if
  - The literals corresponding to u, v are not in the same clause.
  - The literals corresponding to u, v are not negations of each other.

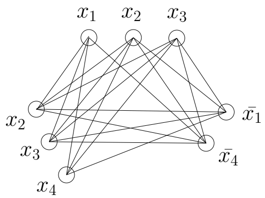
Example 1

Consider formula $F = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\bar{x_1} \vee \bar{x_4})$

First step: create vertices



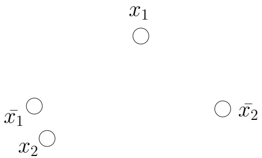
Second step: create edges



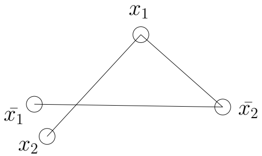
Example 2

Consider formula $F = (x_1) \wedge (\bar{x_1} \vee x_2) \wedge (\bar{x_2})$

First step: create vertices



Second step: create edges



Claim 1: If F has a truth assignment, then G has a k-clique.

Proof: Every clause has a literal set to 1 in the truth assignment. Pick such a literal from every clause. Clearly, no two literals can be negations of each other (because x and $\bar{x}$ cannot both be 1).

Let v_i be the vertex in G corresponding to that literal in the i-th clause ($1 \leq i \leq k$). The claims follows from the fact that there is an edge between any two v_i, v_j for $1 \leq i < j \leq k$. □

Claim 2: If G has a k-clique, F has a truth assignment.

Proof: Let $v_1, v_2, ..., v_k$ be the vertices of the k-clique in G. Their corresponding literals must come from different clauses (because no edge exists between the vertices of two literals from the same clause). Furthermore, the literals corresponding to $v_1, v_2, ..., v_k$ cannot be negations of each other (because no edge exists between the vertices of two literals that are negations of each other). We can therefore construct a truth assignment by setting those k literals to 1. $\square$

The construction of G clearly can be done in polynomial time. We can therefore apply the algorithm $\mathcal{A}$ to determine whether G has a k-clique, and thereby, decide whether a truth assignment exists for F.

This shows that the clique decision problem is NP-hard.

Combining this with the obvious fact that the problem is in $\mathtt{NP}$, we conclude that the problem is NP-complete.