Single Source Shortest Paths with Negative Weights

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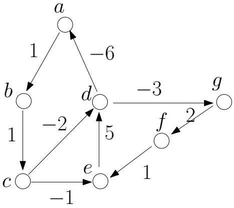
In this lecture, we will continue our discussion on the single source shortest path (SSSP) problem, but this time we will allow the edges to take **negative** weights.

In this case, Dijkstra's algorithm no longer works.

We will learn another algorithm — called **Bellman-Ford's algorithm** — to compute the shortest paths correctly.

Let $G = (V, E)$ be a directed graph. Let w be a function that maps each edge in E to an integer, **which can be positive, 0, or negative**.



Weighted Graphs

Consider a path in G: $(v_1, v_2), (v_2, v_3), ..., (v_\ell, v_{\ell+1})$, for some integer $\ell \geq 1$. We define the **length** of the path as

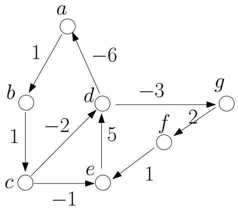
$$\sum_{i=1}^{\ell} w(v_i, v_{i+1}).$$

Given two vertices $u, v \in V$, a **shortest path** from u to v is a path that has the minimum length among all the paths from u to v. Denote by $spdist(u, v)$ the length of the shortest path from u to v.

If v is unreachable from u, then $spdist(u, v) = \infty$.

New: it is possible for $spdist(u, v)$ to be negative.

Example



The path $c \to d \to g$ has length -5.

What is the the shortest path from a to c? Counter-intuitively, it has an **infinite** number of edges! Observe that $spdist(a, c) = -\infty$!

- Why? Because there is a **negative cycle** $a \to b \to c \to d \to a$!

Negative cycle

A path $(v_1, v_2), (v_2, v_3), ..., (v_\ell, v_{\ell+1})$ is a **cycle** if $v_{\ell+1} = v_1$.

It is a **negative cycle** if its length is negative, namely:

$$\sum_{i=1}^{\ell} w(v_i, v_{i+1}) < 0$$

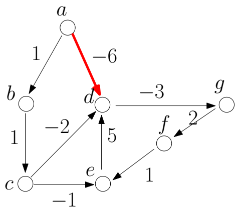
Problem (SSSP): Let $G = (V, E)$ be a directed weighted graph where the weight of every edge can be a positive integer, 0, or a negative integer. **It is guaranteed that G has no negative cycles.** Given a vertex s in V, we want to find, for **every** other vertex $t \in V \setminus \{s\}$, a shortest path from s to t, unless t is unreachable from s.

We will learn an algorithm called **Bellman-Ford's algorithm** that solves both problems in $O(|V||E|)$ time.

We will focus on **computing** $spdist(s, v)$, namely, the shortest path distance from the source vertex s to every other vertex $v \in V \setminus \{s\}$.

Constructing the shortest paths is a bi-product of our algorithm, is easy, and will be left to you.

Example



This graph has no negative cycles.

Lemma: For every vertex $v \in V$, there is a shortest path from s to v that is a **simple path**, namely, a path where no vertex appears twice.

The proof is simple and left to you — note that you must use the condition that no negative cycles are present.

Corollary: For every vertex $v \in V$, there is a shortest path from s to v that has at most $|V| - 1$ edges.

Edge Relaxation

At all times, we will remember, for every $v \in V \setminus \{s\}$, a value $dist(v)$, which records the distance of the shortest path from the source vertex s to u we have found **so far**.

Relaxing an edge (u, v) means:

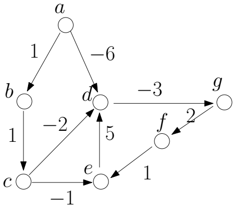
- If $dist(v) < dist(u) + w(u, v)$, do nothing;
- Otherwise, reduce $dist(v)$ to $dist(u) + w(u, v)$.

Bellman-Ford's algorithm

1. Set $dist(s) = 0$, and $dist(v) = \infty$ for all other vertices $v \in V$
2. Repeat the following $|V| - 1$ times
   - Relax all edges in E (the ordering by which the edges are relaxed does not matter)

Example

Suppose that the source vertex is a.



vertex v	$dist(v)$
a	0
b	∞
c	∞
d	∞
e	∞
f	∞
g	∞

Although the edge-relaxation ordering does not matter, for illustration purposes we will relax the edges in **alphabetic** order, namely, (u_1, v_1) before (u_2, v_2) when

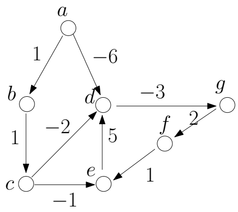
- $u_1 < u_2$ or
- $u_1 = u_2$ but $v_1 < v_2$.

Here is the alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (a, b):



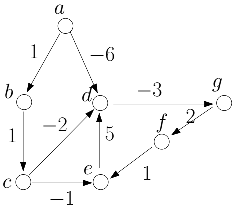
vertex v	$dist(v)$
a	0
b	1
c	∞
d	∞
e	∞
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (a, d):



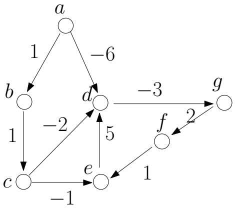
vertex v	$dist(v)$
a	0
b	1
c	∞
d	-6
e	∞
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (b, c):



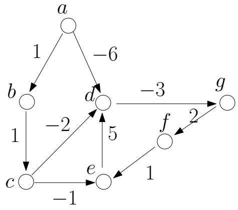
vertex v	$dist(v)$
a	0
b	1
c	2
d	-6
e	∞
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (c, d):



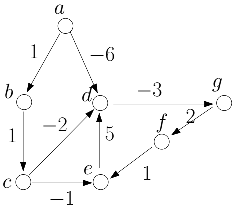
vertex v	$dist(v)$
a	0
b	1
c	2
d	-6
e	∞
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (c, e):



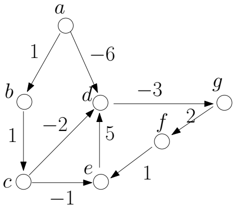
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	1
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (d, g):



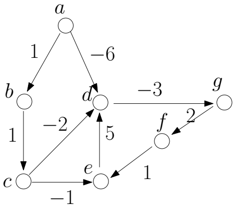
vertex v	$dist(v)$
a	0
b	1
c	2
d	-6
e	1
f	∞
g	-9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (e, d):



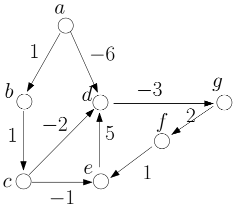
vertex v	$dist(v)$
a	0
b	1
c	2
d	-6
e	1
f	∞
g	-9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (f, e):



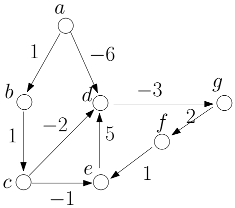
vertex v	$dist(v)$
a	0
b	1
c	2
d	-6
e	1
f	∞
g	-9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges the **first time**.

Here is what happens after relaxing (g, f):



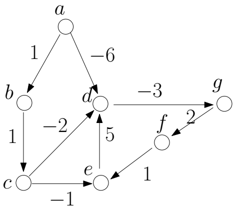
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	1
f	7
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

In the same fashion, relaxing all edges for a **second time**.

Here is the content of the table at the end of this relaxation round:



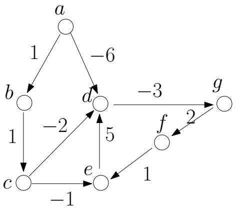
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	1
f	−7
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

In the same fashion, relaxing all edges for a **third time**.

Here is the content of the table at the end of this relaxation round:



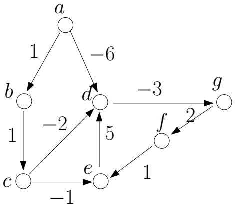
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	−6
f	−7
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

In the same fashion, relaxing all edges for a **fourth time**.

Here is the content of the table at the end of this relaxation round (no changes from the previous round):

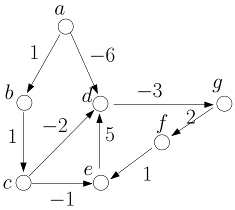


vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	−6
f	−7
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

In the same fashion, relaxing all edges for a **fifth time**, and then a **sixth time**. No more changes to the table:



vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	−6
f	−7
g	−9

The algorithm then terminates here with the above values as the final shortest path distances.

Remark: We did 6 rounds because the purpose is to follow the algorithm description faithfully. In reality, we can stop as soon as no changes are made to the table after some round.

Time

The running time is clearly $O(|V||E|)$.

Correctness

Theorem: Consider any vertex v; suppose that there is a shortest path from s to v that has ℓ edges. Then, after ℓ rounds of edge relaxations, it must hold that $dist(v) = spdist(v)$.

Proof:

We will prove the theorem by induction on ℓ. If $\ell = 0$, then $v = s$, in which case the theorem is obviously correct. Next, assuming the statement's correctness for $\ell < i$ where i is an integer at least 1, we will prove it holds for $\ell = i$ as well.

Denote by π the shortest path from s to v, namely, π has i edges. Let p be the vertex right before v on π.

By the inductive assumption, we know that $dist(p)$ was already equal to $spdist(v)$ after the $(i-1)$-th round of edge relaxations.

In the i-th round, by relaxing edge (p, v), we make sure:

$$\begin{aligned} dist(v) &\leq dist(p) + w(p, v) \\ &= spdist(p) + w(p, v) \\ &= spdist(v). \end{aligned}$$

$\square$