CSCI3160: Regular Exercise Set 5

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Problem 1. Let $G = (V, E)$ be a connected undirected graph where every edge carries a positive integer weight. Divide V into arbitrary disjoint subsets $V_1, V_2, ..., V_t$ for some $t \geq 2$, namely, $V_i \cap V_j = \emptyset$ for any $1 \leq i < j \leq t$, and $\bigcup_{i=1}^{t} V_i = V$. Define an edge $\{u, v\}$ in E a *cross edge* if u and v are not in the same subset (i.e., there is no $i \in [1, t]$ satisfying $u \in V_i$ and $v \in V_i$). Prove: the lightest cross edge must belong to a minimum spanning tree (MST).

Problem 2* (Kruskal's Algorithm). Let $G = (V, E)$ be a connected undirected graph where every edge carries a positive integer weight. Prove that the following algorithm finds an MST of G correctly:

algorithm
1. $S = \emptyset$
2. **while** $|S| < |V| - 1$
3. find the lightest edge $e \in E$ that does not introduce any cycle with the edges in S
4. add e to S
5. the edges in S now form an MST

Problem 3. Consider Σ as an alphabet. Recall that a *code tree* on Σ as a binary tree T satisfying both conditions below:

- C_1 : Every leaf node of T is labeled with a distinct letter in Σ; conversely, every letter in Σ is the label of a distinct leaf node in T.
- C_2 : For every internal node of T, its left edge (if exists) is labeled with 0, and its right edge (if exists) with 1.

Define an *encoding* as a function f that maps each letter $\sigma \in \Sigma$ to a non-empty bit string, which is called the *codeword* of σ. T *produces* an encoding where the code word of a letter $\sigma \in \Sigma$ can be obtained by concatenating the bit labels of the edges on the path from the root to the leaf σ.

Prove:

- The encoding produces by a code tree T is a prefix code.
- Every prefix code is produced by a code tree T.

Problem 4. Consider the alphabet $\Sigma = \{1, 2, ..., n\}$ for some integer $n \geq 1$. Suppose that the frequency of i is *strictly higher than* the frequency of $i + 1$, for any $i \in [1, n - 1]$. Prove: in an optimal prefix code, for any $i \in [1, n - 1]$, the codeword of i cannot be longer than that of $i + 1$.