CSCI3160: Regular Exercise Set 10

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Problem 1. Let $G = (V, E)$ be a weighted directed graph where the edge weights are given by the function $w : E \to \mathbb{Z}$; there are no negative cycles in G. Recall that Johnson's algorithm adds a vertex v_{dummy} to G, and computes the shortest path distance $spdist(v_{dummy}, v)$ from v_{dummy} to every vertex. Then, the weight of each edge (u, v) is modified to:

$$w'(u,v) = w(u,v) + spdist(v_{dummy}, u) - spdist(v_{dummy}, v).$$

Prove: $w'(u, v) \geq 0$.

Problem 2 (Textbook Exercise 24.1-3). Let $G = (V, E)$ a weighted directed graph that does not have negative cycles. Denote by s a vertex in V. Suppose that, for every vertex $v \in V$, there is a shortest path from s to v that has no more than L edges, where L is an integer at most $|V| - 1$. Design an algorithm to find the shortest paths from s to all the other vertices in $O(|E| \cdot L)$ time.

Problem 3 (Single *Sink* Shortest Paths). Let $G = (V, E)$ a weighted directed graph that does not have negative cycles. Denote by t a vertex in V. Design an algorithm to find the shortest path from every vertex $v \in V$ to t. Your algorithm must terminate in $O(|V||E|)$ time.

Problem 4 (Dynamic Programming Nature of Bellman-Ford's). Let $G = (V, E)$ a weighted directed graph that does not have negative cycles. Denote by s a vertex in V. If a path from s to some vertex $v \in V$ uses at most $\ell \in [0, |V| - 1]$ edges, we call it an *ℓ-path* from s to v. Given a vertex v and an integer $\ell \in [0, |V| - 1]$, define $spdist(s, v \mid \ell)$ as the smallest length of all the ℓ-paths from s to v. Prove: for $\ell \geq 1$, it holds that

$$spdist(s, v \mid \ell) = \min \begin{cases} spdist(s, v \mid \ell - 1) \\ \min_{u \in IN(v)} spdist(s, u \mid \ell - 1) + w(u, v) \end{cases} \tag{1}$$

where $IN(v)$ is the set of in-neighbors of v (namely, $u \in IN(v)$ if (u, v) is an edge in E).