Yonsei University Intensive Course
Gravitational Waves
Exercise Set 1

Tjonnie Li

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I Curvature in spherical coordinates

A general displacement vector in the spherical coordinates can be written as

$$\mathbf{r} = r\sin\theta\cos\phi\mathbf{e}_x + r\sin\theta\sin\phi\mathbf{e}_y + r\cos\theta\mathbf{e}_z. \tag{1}$$

(I.1) Determine the three natural basis vectors $\mathbf{e}_r = \partial\mathbf{r}/\partial r$, $\mathbf{e}_\theta = \partial\mathbf{r}/\partial\theta$ and $\mathbf{e}_\phi = \partial\mathbf{r}/\partial\phi$ and give the length of each vector.

(I.2) Write down the line element $d\mathbf{s}$ in the three natural basis vectors of the spherical coordinate system

We now restrict ourselves to the surface of the sphere and impose the conditions $r = c$ where c is some constant, so that our coordinates are $(x^1, x^2) = (\theta, \phi)$. The line element of the sphere's surface can be expressed in terms of the natural basis vectors

(I.3) Write down the line element and the quadratic line element.

(I.4) Determine the metric tensor and its inverse.

(I.5) Calculate all Christoffel symbols $\Gamma^k{}_{ij}$ on the surface of the sphere.

(I.6) How many independent components does the Riemann tensor have? Tip: Use the symmetry properties of the Riemann tensor.

(I.7) Calculate the components of the Riemann tensor.

(I.8) Calculate the components of the Ricci tensor.

(I.9) Calculate the components of the Ricci scalar.

II The Schwarzschild spacetime – Black Holes

In Schwarzschild coordinates, the black hole metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{2}$$

The metric appears to be singular at $r = 2M$, even though we know that observers can move through it, in finite proper time. The horizon is just a coordinate singularity. One set of coordinates that are better suited to describe what happens close to the black hole are the *Kruskal-Szekeres* coordinates.

(II.1) Define coordinates u and v by:

$$u = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M},$$
$$v = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M},$$

for $r > 2M$, and

$$u = \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M},$$
$$v = \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M},$$

for $r < 2M$. Show that the metric in these coordinates is

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} (dv^2 - du^2) + r^2(u,v) \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \tag{3}$$

where $r(u,v)$ is now not a coordinate but a function of u and v determined implicitly by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = u^2 - v^2. \tag{4}$$

(II.2) Draw a diagram in which the horizontal axis is the u axis, the vertical axis is the v axis, and θ and ϕ are suppressed. What is the meaning of a point on such a diagram? Show that radial null geodesics (i.e., lines with $ds = 0$ and also $d\theta = d\phi = 0$) are at 45 degrees to the u and v axes.

(II.3) On the basis of Eq. (4), draw a few representative curves of constant r, for $r < 2M$ as well as for $r > 2M$. Also draw the curves $r = 0$ (the singularity) and $r = 2M$ (the horizon).

(II.4) Draw a few representative curves of constant t.

(II.5) Draw the wordline of an observer who starts on the outside and falls into the black hole. Also draw the worldline of an observer who starts on the outside, and *stays* outside the black hole.

III The FLRW Universe – Cosmology

One can show that the general homogeneous and isotropic spacetime with spatially flat geometry is described by the so-called FLRW metric

$$ds^2 = -d\tau^2 + a^2(\tau)\,(dx^2 + dy^2 + dz^2). \tag{5}$$

(III.1) Compute the Christoffel symbols. Plug these into the expression for the Ricci tensor:

$$R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} = \partial_\nu \Gamma^\nu_{\mu\rho} - \partial_\mu \Gamma^\nu_{\nu\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\nu_{\alpha\mu}. \tag{6}$$

Contract to get the Ricci scalar $R = R^\mu{}_\mu$. Construct the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{7}$$

(III.2) Write the components of the energy-momentum tensor for a perfect fluid,

$$T_{\mu\nu} = \rho u_\mu u_\nu + P\,(g_{\mu\nu} + u_\mu u_\nu). \tag{8}$$

(III.3) Show that the Einstein equations *with cosmological constant*,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{9}$$

reduce to

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\rho + \frac{\Lambda}{3}, \tag{10}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}. \tag{11}$$

(III.4) Now set $\Lambda = 0$. Combine Eqns. (10) and (11) to find

$$\dot{\rho} = -3(\rho + P)\,\frac{\dot{a}}{a}. \tag{12}$$

Hence this equation follows from the Einstein equations. However, show that it already follows from energy-momentum conservation,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{13}$$

(III.5) Assume pressureless dust ($P = 0$). Integrate Eq. (12) to find ρ as a function of a, up to an integration factor ρ_0. Substitute the solution into Eq. (10) – still assuming $\Lambda = 0$ – and solve for $a(\tau)$. Check that your result is consistent with Eq. (11).

(III.6) Now assume radiation ($P = \rho/3$). Again integrate Eq. (12) to find $\rho(a)$. Explain why at an early stage in the evolution of the Universe, radiation would have been the dominant factor in the dynamics of the Universe. Then again solve Eq. (10) for $a(\tau)$. Discuss the difference with the solution for pressureless dust.

(III.7) Discuss qualitatively what would have been different in (II.e) and (II.f) had you set $\Lambda > 0$ or $\Lambda < 0$.