Intensive Course in Physics
Gravitational Waves

Tjonnie G. F. Li

Chapter 2: Properties of Gravitational Waves

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Instanteneous Forces

- In Newton's theory of gravity, any changes in the distribution of matter are felt instantaneously at arbitrarily large distances.
- Governed by the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho \tag{1}$$

- Considered unsatisfactory already by some of his contemporaries in the late 17th century.
- Prominent scientists (*e.g.* Laplace) tried to come up with some dynamical mechanism
- Even bigger problem when special relativity (1905) was introduced
  - Strict speed limit on communication of any kind

Electromagnetism

- Maxwell's theory of electromagnetism does not have instantaneous action at a distance.
- **E** and **B** at a distance r from the source depend on what the source was doing at a time $t - r/c$.
- The time lag, r/c, is the time needed for a signal to cross the distance r if it traveled at the speed of light: electromagnetism obeys Einstein's speed limit.
- **E** and **B** obey a wave equation

$$\left(c^2\nabla^2 - \partial_t^2\right)\mathbf{E} = 0 \tag{2}$$

$$\left(c^2\nabla^2 - \partial_t^2\right)\mathbf{B} = 0 \tag{3}$$

- Changes in a charge/current distribution are communicated to the rest of space by electromagnetic waves.

Electromagnetic field does not just "track" its sources; it has dynamics of its own.

General Relativity

- After special relativity was developed it was soon speculated that the gravitational field might also be dynamical.
- Changes in the gravitational field should propagate in a wave-like fashion, no faster than the speed of light
- Eliminating instantaneous action at a distance.
- General theory of relativity of 1916 indeed incorporated all these ideas.

General Relativity predicts the existence of gravitational waves

Weak Fields

- Study GWs in the regime where gravitational fields are weak.
- Write spacetime metric $g_{\mu\nu}$ as the Minkowski spacetime $\eta_{\mu\nu}$ plus a small correction $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{4}$$

- Write the Einstein equations to first order in $h_{\mu\nu}$,

Coordinate transforms I

- Einstein Field equations are invariant under general coordinate transformations,

$$x^\mu \longrightarrow x'^\mu(x), \tag{5}$$

- Metric transforms as

$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'_\nu} g_{\rho\sigma}(x). \tag{6}$$

- This invariance is broken when we choose a fixed background $\eta_{\mu\nu}$ as in Eq. (4)
- Instead, we look for a specific reference frame where Eq. (4) holds in a sufficiently large region of spacetime.
- No longer be able to transform the metric at will.

Coordinate transforms II

- Still exists a (much more limited) family of transformations which respects our choice of frame
- Consider the following gauge transformations

$$x^{\mu} \longrightarrow x'^{\mu} = x^{\mu} + \xi^{\mu}(x), \tag{7}$$

  - where $|\partial_{\rho}\xi_{\mu}|$ are at most of the same order as $|h_{\mu\nu}|$
- Substituting into the transformation law of the metric, Eq. (6) and keeping only lowest-order terms

$$h_{\mu\nu}(x) \longrightarrow h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}). \tag{8}$$

Coordinate transforms III

- We can also perform global (x-independent) Lorentz transformations,

$$x^\mu \longrightarrow x'^\mu = \Lambda^m{}_\nu x^\nu. \tag{9}$$

- $h_{\mu\nu}$ transforms as

$$h'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x). \tag{10}$$

- $h_{\mu\nu}$ is a tensor under Lorentz transformations, as long as one keeps $|h_{\mu\nu}| \ll 1$

Linearised Einstein's Field Equations I

- To leading order in $h_{\mu\nu}$, the Riemann tensor is

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\right). \quad (11)$$

- Linearized Riemann tensor is invariant under the gauge transformations Eq. (8)

Linearised Einstein's Field Equations II

- It will be convenient to introduce

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{12}$$

  - where $h = \eta^{\mu\nu}h_{\mu\nu}$
- Note that $\bar{h} \equiv \eta^{\mu\nu}h_{\mu\nu} = h - 2h = -h$ so that

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \tag{13}$$

Linearised Einstein's Field Equations III

- Using Eq. (11), and Eq. (13), the linearized Einstein equations take the form

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \quad (14)$$

  - where $\Box \equiv \partial_\mu\partial^\mu$ is the usual d'Alembertian.

Linearised Einstein's Field Equations IV

- Use residual gauge freedom Eq. (7) to further simplify
- $\bar{h}_{\mu\nu}$ transforms as

$$\bar{h}_{\mu\nu} \longrightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho). \tag{15}$$

- Impose the harmonic gauge

$$\partial^\nu \bar{h}_{\mu\nu} = 0. \tag{16}$$

- Last three terms in the LHS of Eq. (14) vanish

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \tag{17}$$

These are the linearized Einstein equations.

Linearised Einstein's Field Equations V

- Note that our ability to impose the harmonic gauge Eq. (16) Eq. (17) implies that

$$\partial^\nu T_{\mu\nu} = 0. \tag{18}$$

- In the full theory one has $\nabla^\nu T_{\mu\nu}$ with ∇^ν the covariant derivative

Vacuum Solutions I

- The general solution to the linearized Einstein equations at $(t, \mathbf{x})$ is

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = -4\frac{G}{c^2}\int_{\mathcal{V}} \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \qquad (19)$$

- Unlike the Newtonian potential, the value of $\bar{h}_{\mu\nu}$ at a point $\mathbf{x}$ arbitrarily far from the source $\mathcal{S}$ does not have instantaneous knowledge of what happens at at $\mathcal{V}$.
- There are time lags $|\mathbf{x} - \mathbf{x}'|/c$, these being the times needed for a signal traveling at the speed of light to get from points $\mathbf{x}'$ inside the source to the point $\mathbf{x}$. Just like electromagnetism

gravity does not have instantaneous action at at distance after all.

Vacuum Solutions II

- Outside the source $T_{\mu\nu} = 0$, and Eq. (17) reduces to

$$\Box \bar{h}_{\mu\nu} = 0, \tag{20}$$

- or written in full

$$\left(-\frac{1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = 0. \tag{21}$$

This is just a wave equation, for waves traveling at the speed of light

Vacuum Solutions III

- Solutions can be written as superpositions of plane waves with frequencies ω and wave vectors $\mathbf{k}$,

$$A_{\mu\nu} \cos(\omega t - \mathbf{k} \cdot \mathbf{x}), \tag{22}$$

  - where $\omega = c|\mathbf{k}|$, and $A_{\mu\nu}$ has constant components.

Newtonian Limit I

- For weak gravitational fields and small velocities,

$$|T^{00}| \gg |T^{i0}| \gg |T^{ii}| \tag{23}$$

- which translates into

$$|\bar{h}^{00}| \gg |\bar{h}^{i0}| \gg |\bar{h}^{ii}| \tag{24}$$

- In this regime,

$$T^{00}/c^2 \simeq \rho \tag{25}$$

- The equation Eq. (17) then reduces to

$$\Box \bar{h}^{00} \simeq -\frac{16\pi G}{c^2}\,\rho. \tag{26}$$

Newtonian Limit II

- For sources moving with 3-velocity v such that $v/c \ll 1$, $(1/c^2)\partial^2\bar{h}^{00}/\partial t^2$ is of order $(v/c)^2\,\partial^2\bar{h}^{00}/\partial(x^i)^2$,
- Eq. (21) reduces to

$$c^2\nabla^2\bar{h}^{00} \simeq -16\pi G\rho. \tag{27}$$

- With the identification

$$c^2\bar{h}^{00} = -4\phi, \tag{28}$$

- this becomes

$$\nabla^2\phi = 4\pi G\rho, \tag{29}$$

Poisson equation for the gravitational potential ϕ in Newton's theory of gravity.

Newtonian Limit III

- The identification Eq. (28) is consistent with the motion of point particles in the weak-field, low-velocity regime.
- In general relativity this motion is governed by the geodesic equation,

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{30}$$

- Recall that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$.
- For $v/c \ll 1$, the proper time τ will approximately coincide with the coordinate time t associated with the background spacetime $\eta_{\mu\nu}$.
- Moreover, $dx^0/dt \simeq c$ while $dx^i/dt = \mathcal{O}(v)$.

Newtonian Limit IV

- Hence, to leading order we need only retain the term in Eq. (30) with $\mu = \nu = 0$

$$\frac{d^2x^i}{dt^2} \simeq -c^2\Gamma^i_{00} = c^2\left(\frac{1}{2}\partial^i h_{00} - \partial_0 h_0{}^i\right). \quad (31)$$

- For a non-relativistic source, the time derivative is again of higher order than the spatial derivatives

$$\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\partial^i h_{00}. \quad (32)$$

- This is an equation in terms of h_{00} rather than $\bar{h}_{00}$.

Newtonian Limit V

- Since $\bar{h}^{00}$ dominates all other components of $\bar{h}^{\mu\nu}$,

$$h = h^{\mu}{}_{\mu} = -\bar{h}^{\mu}{}_{\mu} = \bar{h}^{00}, \tag{33}$$

- From Eq. (13) and Eq. (28) we get

$$c^2 h_{00} = -2\phi. \tag{34}$$

- Substituting this into Eq. (32) we retrieve Newton's second law for a force with potential ϕ:

$$\mathbf{a} = -\nabla\phi, \tag{35}$$

  - with $\mathbf{a}$ being the acceleration 3-vector.

Retrieved both Newton's equation for the gravitational potential Eq. (29), and the Newtonian motion of a particle in such a potential Eq. (35).

Newtonian Limit VI

- The most general solution of Eq. (29) is

$$\phi(t, \mathbf{x}) = G \int_{\mathcal{V}} \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \tag{36}$$

- The fact that $\rho(t, \mathbf{x}')$ in the integrand does not include a time lag $|\mathbf{x} - \mathbf{x}'|/c$ is due to the absence of a double time derivative in Eq. (29)
- In Eq. (27) this term could be neglected because $v/c \ll 1$.

Degrees of Freedom I

- *A priori*, $\bar{h}_{\mu\nu}$ has 10 independent components
- Some are gauge artefact and can be eliminated by using transformations of the form Eq. (15).
- Harmonic gauge Eq. (16) eliminates 4 components
- This gauge choice still allows for residual freedom.

Degrees of Freedom II

- Condition Eq. (16) is not spoiled by a transformation Eq. (15)

$$\Box \xi_\mu = 0. \tag{37}$$

- Note that if $\Box \xi_\mu = 0$ then also $\Box \xi_{\mu\nu} = 0$, where

$$\xi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho, \tag{38}$$

  - because $\Box$ commutes with ∂_μ.

We can use 4 functions $\xi_\mu(x)$ to eliminate 4 more components of $\bar{h}_{\mu\nu}$ without spoiling either the harmonic gauge or the simple form of the linearized Einstein equations (17).

TT-Gauge I

- We can choose $\xi_0(x)$ such that the trace

$$\bar{h} = 0 \tag{39}$$

- such that

$$\bar{h}_{\mu\nu} = h_{\mu\nu} \tag{40}$$

- Furthermore, we can choose the three functions $\xi_i(x)$, $i = 1, 2, 3$ so that

$$h_{0\mu}(x) = 0. \tag{41}$$

- From Eq. (40) the harmonic gauge condition with $\mu = 0$ then becomes

$$\partial^0 h_{00} + \partial^i h_{0i} = 0. \tag{42}$$

TT-Gauge II

- Since we just set $h_{0i} = 0$, this reduces to

$$\partial^0 h_{00} = 0, \tag{43}$$

- so that h_{00} does not depend on time.
- A time-independent contribution to h_{00} corresponds to the static part of the gravitational interaction, i.e., to the Newtonian potential of the source arising from its total mass without contributions due to motion.
- The gravitational wave is the time-dependent part, and since this is our focus here we will just set $h_{00} = 0$.
- Strictly speaking we should retain the Newtonian contribution h_{00}, but it will have no effect on gravitational wave detection

TT-Gauge III

- The spatial part of the harmonic gauge (with $\mu = i = 1, 2, 3$) is then

$$\partial^j h_{ij} = 0, \tag{44}$$

- and the condition $h = 0$ becomes

$$h^i{}_i = 0 \tag{45}$$

- In summary, we have

$$h_{0\mu} = 0 \tag{46}$$
$$h^i{}_i = 0 \tag{47}$$
$$\partial^j h_{ij} = 0. \tag{48}$$

TT-Gauge IV

- Used up all of our gauge freedom and are left with two degrees of freedom.
- The gauge in which the conditions Eq. (48) hold is called the *transverse-traceless gauge*, or TT gauge.
- The metric perturbation in the TT gauge is denoted h_{ij}^{TT}.

TT-Gauge V

- Eq. (21) has plane wave solutions of the form

$$h_{ij}^{\mathrm{TT}} = e_{ij}(\mathbf{k}) \cos(k_\mu x^\mu), \tag{49}$$

  - with $k_\mu = (\omega/c, \mathbf{k})$, and $\omega = c|\mathbf{k}|$.
- The tensor $e_{ij}(\mathbf{k})$ is called the polarization tensor.
- For a single plane wave with wave vector $\mathbf{k}$, the condition $\partial^j h_{ij} = 0$ becomes

$$\mathbf{k}^j h_{ij}^{\mathrm{TT}} = 0 \qquad n^j h_{ij}^{\mathrm{TT}} = 0 \tag{50}$$

  - where $\hat{\mathbf{n}} = \mathbf{k}/|\mathbf{k}|$ is the unit vector in the direction of motion.

Non-zero components of h_{ij}^{T} are in the plane that is transverse to $\hat{\mathbf{n}}$.

TT-Gauge VI

- Suppose we choose the z axis to lie in the direction of $\hat{\mathbf{n}}$.
- Taking into account symmetry, transversality and tracelessness of h_{ij}^{TT}, we get

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega(t - z/c)\right] \tag{51}$$

- In terms of the line element ds^2, we have

$$\begin{aligned} ds^2 = & -c^2 dt^2 + dz^2 + \left[1 + h_+ \cos[\omega(t - z/c)]\right] dx^2 \\ & + \left[1 - h_+ \cos[\omega(t - z/c)]\right] dy^2 + 2h_\times \cos[\omega(t - z/c)]\, dx dy. \end{aligned} \tag{52}$$

Geodesic Deviation I

What is the effect of the perturbation h on matter?

- Consider the relative motion of two nearby test particles in free fall.
- A free-falling test particle obeys the geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{53}$$

  - where τ is proper time.

Geodesic Deviation II

- Now consider two nearby free-falling particles, at $x^\mu(\tau)$ and $x^\mu(\tau) + \zeta^\mu$.
- The first particle is subject to Eq. (53) while the second one obeys

$$\frac{d^2(x^\mu + \zeta^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x+\zeta)\frac{d(x^\nu + \zeta^\nu)}{d\tau}\frac{d(x^\rho + \zeta^\rho)}{d\tau} = 0. \tag{54}$$

- Taking the difference between Eq. (54) and Eq. (53)
- Expanding to first order in ζ^μ

$$\frac{d^2\zeta^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\zeta^\rho}{d\tau} + \zeta^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{55}$$

Geodesic Deviation III

- Introduce the covariant derivative of a vector field V^μ along the curve $x^\mu(\tau)$:

$$\frac{DV^\mu}{D\tau} = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} V^\nu \frac{dx^\rho}{d\tau}. \tag{56}$$

- Using this and the definition of the Riemann tensor, recast Eq. (55) as

$$\frac{D^2\zeta^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}\zeta^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}. \tag{57}$$

This is the equation of geodesic deviation, which expresses the relative motion of nearby particles in terms of a tidal force determined by the Riemann tensor.

Geodesic Deviation IV

- Given a point P along a geodesic, there always exists a coordinate transformation that will make the Christoffel symbols vanish at P:

$$\Gamma^{\mu}_{\nu\rho}(P) = 0. \tag{58}$$

- This is just the Local Lorentz Frame
- Furthermore, let us consider particles which move non-relativistically,
- *i.e.* spatial motion $dx^i/d\tau$ is negligible compared to $dx^0/d\tau$.
- Eq. (55) becomes

$$\frac{d^2\zeta^i}{d\tau^2} + \zeta^\sigma \partial_\sigma \Gamma^i_{00} \left(\frac{dx^0}{d\tau}\right)^2 = 0. \tag{59}$$

Geodesic Deviation V

- Quantity $\partial_\sigma \Gamma^i_{00}$ is evaluated at the point P, i.e., at $x^i = 0$,
- Metric is of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^i x^j) \tag{60}$$

- Therefore

$$\zeta^\sigma \partial_\sigma \Gamma^i_{00} = \zeta^j \partial_j \Gamma^i_{00} \tag{61}$$

- Since at P both $\Gamma^\mu_{\nu\rho} = 0$ and $\partial_0 \Gamma^i_{0j} = 0$,
- One has

$$R^i{}_{0j0} = \partial_j \Gamma^i_{00} - \partial_0 \Gamma^i_{0j} = \partial_j \Gamma^i_{00} \tag{62}$$

Geodesic Deviation VI

- Finally

$$\frac{d^2\zeta^i}{d\tau^2} = -R^i{}_{0j0}\zeta^j \left(\frac{dx^0}{d\tau}\right)^2 . \tag{63}$$

- If the test masses are moving non-relativistically then $dx^0/d\tau \simeq c$ and $\tau = t$
- We finally arrive at

$$\ddot{\zeta}^i = -c^2 R^i{}_{0j0}\zeta^j , \tag{64}$$

  - where a dot denotes derivation with respect to t.

Riemann Tensor

- In the linearized theory, Riemann tensor is *invariant*
- Evaluating (11) in the TT frame we get

$$R^{i}{}_{0j0} = R_{i0j0} = -\frac{1}{c^2}\ddot{h}_{ij}^{\mathrm{TT}}. \tag{65}$$

- Hence, at the point P, the geodesic deviation equation reduces to

$$\ddot{\zeta}^{i} = \frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}}\zeta^{j}. \tag{66}$$

Waves I

- Monochromatic gravitational wave propagating in the z-direction
- Study its effect on test particles in the (x, y) plane.
- Focus on the $+$ polarization.
- At $z = 0$ and choosing the origin of time such that $h_{ij}^{\mathrm{TT}} = 0$ at $t = 0$,

$$h_{ij}^{\mathrm{TT}} = h_+ \sin(\omega t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} . \tag{67}$$

Waves II

- Consider a point particle in the (x, y) plane

$$\zeta^i = (x_0 + \delta x(t), y_0 + \delta y(t), 0) \tag{68}$$

  - where (x_0, y_0) is the unperturbed position and $\delta x(t)$, $\delta y(t)$ the displacement caused by the gravitational wave.
- From Eq. (66) and assuming that (x_0, y_0) and $(0, 0)$ are on "nearby" geodesics,

$$\begin{aligned} \delta\ddot{x} &= -\frac{h_+}{2}(x_0 + \delta x)\,\omega^2 \sin(\omega t), \\ \delta\ddot{y} &= +\frac{h_+}{2}(y_0 + \delta y)\,\omega^2 \sin(\omega t). \end{aligned} \tag{69}$$

Waves III

- Assume small displacements compared with the unperturbed position, $\delta x \ll x_0$ and $\delta y \ll y_0$

$$\delta\ddot{x} = -\frac{h_+}{2} x_0\, \omega^2 \sin(\omega t),$$
$$\delta\ddot{y} = +\frac{h_+}{2} y_0\, \omega^2 \sin(\omega t), \tag{70}$$

- which integrates to

$$\delta x(t) = \frac{h_+}{2} x_0\, \omega^2 \sin(\omega t),$$
$$\delta y(t) = -\frac{h_+}{2} y_0\, \omega^2 \sin(\omega t). \tag{71}$$

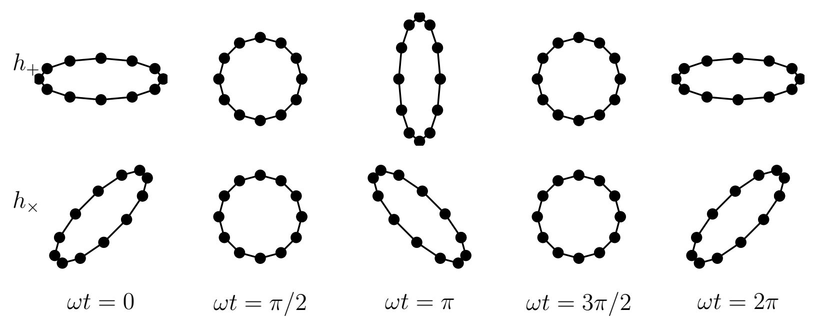
Waves IV

- Completely analogously, for the cross polarization

$$\delta x(t) = \frac{h_\times}{2} y_0 \, \omega^2 \sin(\omega t),$$
$$\delta y(t) = \frac{h_\times}{2} x_0 \, \omega^2 \sin(\omega t). \quad (72)$$

Waves V

Deformation of a ring of test particles



The deformation of a ring of test particles due to the $+$ and $\times$ polarizations.

Higher Order Einstein's Field Equations I

- Linearized Einstein equations in vacuum are

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} = 0, \quad (73)$$

- Where $R^{(1)}_{\mu\nu}$ is the Ricci tensor up to linear terms in the small perturbation $h_{\mu\nu}$ around the flat background $\eta_{\mu\nu}$
- Computed from the linearized Riemann tensor Eq. (11)
- Schematically, the linearized Einstein equations can be written as

$$G^{(1)}_{\mu\nu}[h_{\rho\sigma}] = 0, \quad (74)$$

  - where $G^{(1)}_{\mu\nu}$ is the Einstein tensor to first order in $h_{\mu\nu}$ and its derivatives.

Higher Order Einstein's Field Equations II

- Given a solution $h_{\mu\nu}$ of the linearized Einstein equations, the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ will generally not be a solution to the *full* Einstein equations.
- Does not even solve second order Einstein equations
- Indeed, expanding the Einstein tensor as

$$G_{\mu\nu}[h_{\rho\sigma}] = G^{(1)}_{\mu\nu}[h_{\rho\sigma}] + G^{(2)}_{\mu\nu}[h_{\rho\sigma}] + \dots \tag{75}$$

- where $G^{(2)}_{\mu\nu}$ collects all second order terms
- Typically, $G^{(2)}_{\mu\nu}[h_{\rho\sigma}] \neq 0$.

Higher Order Einstein's Field Equations III

- The second order Einstein equations are

$$G^{(1)}_{\mu\nu}[h_{\rho\sigma}] + G^{(2)}_{\mu\nu}[h_{\rho\sigma}] = 0. \tag{76}$$

- Suppose a solution $h_{\mu\nu}$ of the linearized equations Eq. (74)
- If we have

$$G^{(2)}_{\mu\nu}[h_{\rho\sigma}] \neq 0 \tag{77}$$

- Second order equation Eq. (76) does not hold.

Higher Order Einstein's Field Equations IV

- Correct the second order equation Eq. (76) by adding smaller correction $h^{(2)}_{\mu\nu}$
- These have to satisfy

$$G^{(2)}_{\mu\nu}[h_{\rho\sigma}] + G^{(1)}_{\mu\nu}[h^{(2)}_{\rho\sigma}] = 0. \tag{78}$$

- We can write this in the form

$$G^{(1)}_{\mu\nu}[h^{(2)}_{\mu\nu}] = \frac{8\pi G}{c^4} t_{\mu\nu} \tag{79}$$

- with the identification that

$$t_{\mu\nu} = -\frac{c^4}{8\pi G} G^{(2)}_{\mu\nu}[h_{\rho\sigma}]. \tag{80}$$

Higher Order Einstein's Field Equations V

- The corrected Einstein equations then become

$$G^{(1)}_{\mu\nu}[h_{\rho\sigma} + h^{(2)}_{\rho\sigma}] = \frac{8\pi G}{c^4} t_{\mu\nu}, \tag{81}$$

- where we have

$$t_{\mu\nu} = -\frac{c^4}{8\pi G} G^{(2)}_{\mu\nu}[h_{\rho\sigma}]. \tag{82}$$

- To second order, $h_{\mu\nu}$ causes the same correction to the spacetime metric as would be produced by additional ordinary matter with stress-energy tensor $t_{\mu\nu}$.
- Note that $t_{\mu\nu}$ is symmetric, and if $h_{\mu\nu}$ satisfies the linearized Einstein equations then $\partial^\mu t_{\mu\nu} = 0$, hence it is conserved.

Spatial Averaging I

- It is tempting to regard $t_{\mu\nu}$ as the stress-energy tensor of the gravitational field itself, valid to second order in deviation from flatness.
- However, $t_{\mu\nu}$ is not gauge invariant
- Changes under the transformations Eq. (8).

In general relativity there is no local notion of the energy density of the gravitational field.

Spatial Averaging II

- Evaluating $t_{\mu\nu}$ by averaging it over a small spatial volume surrounding that point
- Obtain a gauge-invariant quantity.

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}\right\rangle, \tag{83}$$

  - where $\langle \ldots \rangle$ denotes the average over a bounded spatial volume

Spatial Averaging III

- Second order contributions to the Ricci tensor are

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\Big[\frac{1}{2}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} + h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\nu\partial_\sigma h_{\rho\mu} - h^{\rho\sigma}\partial_\mu\partial_\sigma h_{\rho\nu}$$
$$h^{\rho\sigma}\partial_\rho\partial_\sigma h_{\mu\nu} + \partial^\sigma h^\rho_\nu \partial_\sigma h_{\rho\mu} - \partial^\sigma h^\rho_\nu \partial_\rho h_{\sigma\mu} - \partial_\sigma h^{\rho\sigma}\partial_\nu h_{\rho\mu}$$
$$+\partial_\sigma h^{\rho\sigma}\partial_\rho h_{\mu\nu} - \partial_\sigma h^{\rho\sigma}\partial_\mu h_{\rho\nu} - \frac{1}{2}\partial^\rho h\partial_\rho h_{\mu\nu} + \frac{1}{2}\partial^\rho h\partial_\nu h_{\rho\mu}$$
$$+\frac{1}{2}\partial^\rho h\partial_\mu h_{\rho\nu}\Big]. \quad (84)$$

- Due to the averaging in Eq. (83), the expression for $t_{\mu\nu}$ will end up being quite simple.
- Discard boundary terms since we assume an integration volume with a boundary
- Time dependence of $h_{\mu\nu}$ will be through a retarded time
- But then $\partial_0 h_{\mu\nu} = -\partial_z h_{\mu\nu}$.

Stress-energy Psuedo Tensor I

- Make all terms in Eq. (84) except for first two vanish using
  - Gauge condition $\partial_\mu h^{\mu\nu} = 0$
  - Tracelessness condition $h = 0$
  - Field equations $\Box h_{\mu\nu} = 0$
- Remaining terms can be combined to get

$$\langle R^{(2)}_{\mu\nu} \rangle = -\frac{1}{4} \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle. \tag{85}$$

- Thus, we arrive at

$$t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle. \tag{86}$$

Stress-energy Psuedo Tensor II

- The change in $t_{\mu\nu}$ under the gauge transformations Eq. (8)

$$\begin{aligned}\delta t_{\mu\nu} &= \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\rho\sigma}\partial_\nu(\delta h^{\rho\sigma}) + \partial_\mu(\delta h_{\rho\sigma})\partial_\nu h^{\rho\sigma}\right\rangle \\ &= \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\rho\sigma}\partial_\nu(\partial^\rho\xi^\sigma + \partial^\sigma\xi^\rho) + (\mu \leftrightarrow \nu)\right\rangle \\ &= \frac{c^4}{16\pi G}\left\langle \partial_\mu h_{\rho\sigma}\partial_\nu\partial^\rho\xi^\sigma + (\mu \leftrightarrow \nu)\right\rangle . \end{aligned} \tag{87}$$

- Inside the average $\langle \ldots \rangle$ we can
  - integrate ∂^ρ by parts
  - use the gauge condition $\partial^\rho h_{\rho\sigma} = 0$.
- Therefore $\delta t_{\mu\nu} = 0$, and $t_{\mu\nu}$ is gauge invariant.

Stress-energy Psuedo Tensor III

- Hence it only depends on the physical content of the spacetime perturbation $h_{\mu\nu}$
- In that gauge, the energy gravitational energy density is

$$t^{00} = \frac{c^2}{32\pi G}\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}} \rangle, \tag{88}$$

  where the dot denotes derivation w.r.t. time; note that $\partial_0 = (1/c)\partial_t$.
- In terms of the two gravitational wave polarizations

$$t^{00} = \frac{c^2}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle. \tag{89}$$

Energy of Gravitational Waves I

- Gravitational energy inside volume V is

$$E_V = \int_V d^3x\, t^{00}. \tag{90}$$

- The gravitational energy going through surface S per unit of time is then given by

$$\frac{dE_{\mathrm{GW}}}{dt} = -\int_V d^3x\, \partial_t t^{00}, \tag{91}$$

- Where the minus sign indicates that we are interested in the energy leaving the surface.

Energy of Gravitational Waves II

- Using conservation of gravitational stress-energy $\partial_\mu t^{\mu\nu} = 0$

$$\frac{1}{c}\frac{dE_{\mathrm{GW}}}{dt} = \int_V d^3x\, \partial_i t^{0i} = \int_S dA\, n_i t^{0i}, \tag{92}$$

  - where dA is the infinitesimal surface element and $\hat{\mathbf{n}}$ the unit normal to S.
- If S is a sphere then
  - Unit vector $\hat{\mathbf{n}} = \hat{r}$
  - $dA = r^2 d\Omega$, with r the sphere's radius
  - $d\Omega = \sin(\theta) d\theta d\phi$ in the usual angular coordinates (θ, ϕ).

Energy of Gravitational Waves III

- One then has

$$\frac{dE_{\rm GW}}{dt} = cr^2 \int d\Omega\, t^{0r}, \tag{93}$$

$$t^{0r} = \frac{c^4}{32\pi G} \left\langle \partial^0 \dot{h}_{ij}^{\rm TT} \partial^r h_{ij}^{\rm TT} \right\rangle. \tag{94}$$

- If r is sufficiently large, a gravitational wave propagating radially outward has the form

$$h_{ij}^{\rm TT} = \frac{1}{r} f_{ij}(t - r/c). \tag{95}$$

- The derivative with respect to r then gives

$$\frac{\partial}{\partial r} h_{ij}^{\rm TT} = -\frac{1}{r^2} f_{ij}(t - r/c) + \frac{1}{r} \frac{\partial}{\partial r} f_{ij}(t - r/c). \tag{96}$$

Energy of Gravitational Waves IV

- Note that

$$\frac{\partial}{\partial r} f_{ij}(t - r/c) = -\frac{1}{c}\frac{\partial}{\partial t} f_{ij}(t - r/c), \tag{97}$$

$$\begin{aligned} \frac{\partial}{\partial r} h_{ij}^{\mathrm{TT}} &= -\partial_0 h_{ij}^{\mathrm{TT}} + \mathcal{O}\left(\frac{1}{r^2}\right) \\ &= +\partial^0 h_{ij}^{\mathrm{TT}} + \mathcal{O}\left(\frac{1}{r^2}\right). \end{aligned} \tag{98}$$

- Hence, at large distances one has $t^{0r} = t^{00}$, and

$$\frac{dE_{\mathrm{GW}}}{dt} = cr^2 \int d\Omega \, t^{00}. \tag{99}$$

Energy of Gravitational Waves V

- Using expression Eq. (88) for the gravitational energy density,

$$\frac{dE_{\rm GW}}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega \, \langle \dot{h}_{ij}^{\rm TT} \dot{h}_{ij}^{\rm TT} \rangle, \qquad (100)$$

- or in terms of the two polarizations,

$$\frac{dE_{\rm GW}}{dt} = \frac{c^3 r^2}{16\pi G} \int d\Omega \, \langle \dot{h}_{+}^2 + \dot{h}_{\times} \rangle. \qquad (101)$$

Thus, gravitational waves carry away energy, which they can deposit into physical systems.

- ravitational waves also carry *momentum*.
- Given a volume V, the gravitational momentum inside it is

$$P^k = \frac{1}{c}\int_V d^3x\, t^{0k}. \tag{102}$$

- Outgoing momentum per unit time is

$$\begin{aligned}\frac{\partial P^k_{\mathrm{GW}}}{dt} &= -\int_V d^3x\, \partial_0 t^{0k} \\ &= r^2 \int_S d\Omega t^{0k}.\end{aligned} \tag{103}$$

- Using Eq. (86) we arrive at

$$\frac{\partial P^k_{\mathrm{GW}}}{dt} = -\frac{c^3 r^2}{32\pi G}\int_S d\Omega\, \langle \dot{h}^{\mathrm{TT}}_{ij} \partial^k h^{\mathrm{TT}}_{ij}\rangle. \tag{104}$$

Green's functions I

- The field equations of linearized gravity are Eq. (17).

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{105}$$

- Since these are linear equations, they can be solved using Green's functions.
- The appropriate Green's function here is the one that solves the equation

$$\Box_x G(x - x') = \delta^4(x - x'), \tag{106}$$

  - where x, x' are any two spacetime points
  - derivatives in the LHS are with respect to the components of $x = (ct, \mathbf{x})$.

Green's functions II

- For a given $T_{\mu\nu}$, the solution to Eq. (105) is

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4}\int d^4x'\, G(x-x')T_{\mu\nu}. \tag{107}$$

- Choosing boundary conditions such that there is no incoming radiation from infinity retarded Green's function

$$G(x-x') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(x^0_{\mathrm{ret}} - x'^0), \tag{108}$$

- Where $x'^0 = ct'$, $x^0_{\mathrm{ret}} = ct_{\mathrm{ret}}$
- Retarded time t_{ret} is given by

$$t_{\mathrm{ret}} = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}. \tag{109}$$

Green's functions III

- Eq. (107) then becomes

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|} T_{\mu\nu}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right). \quad (110)$$

Projection Operator I

- Look for solution in the TT-gauge.
- Let $\hat{\mathbf{n}}$ be the direction of propagation of a gravitational wave.
- Then the following operator removes the component of any spatial vector along the direction $\hat{\mathbf{n}}$:

$$P_{ij} \equiv \delta_{ij} - n_i n_j. \tag{111}$$

- Given a spatial vector v^i, the vector $w^i = P_{ij} v^j$ is transverse:

$$\hat{\mathbf{n}} \cdot \mathbf{w} = n^i P_{ij} v^j = 0. \tag{112}$$

- P_{ij} is a projector:

$$P_{ik} P_{kj} = P_{ij}. \tag{113}$$

Projection Operator II

- Using P_{ij}, we now construct

$$\Lambda_{ij,kl}(\hat{\mathbf{n}}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}. \tag{114}$$

- This is also a projector, in the sense that

$$\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}. \tag{115}$$

- It is transverse in all indices: $n^i\Lambda_{ij,kl} = 0$, $n^j\Lambda_{ij,kl} = 0$
- It is also traceless with respect to the first and last index pairs:

$$\Lambda_{ii,kl} = \Lambda_{ij,kk} = 0. \tag{116}$$

- Finally, it is symmetric under the interchange $(i,j) \leftrightarrow (k,l)$:

$$\Lambda_{ij,kl} = \Lambda_{kl,ij}. \tag{117}$$

Projection Operator III

- The explicit expression for $\Lambda_{ij,kl}$ is:

$$\Lambda_{ij,kl}(\hat{\mathbf{n}}) = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl} + \frac{1}{2} n_i n_j n_k n_l. \tag{118}$$

- The projection is equivalent to performing a gauge transformation that brings $h_{\mu\nu}$ into the TT gauge

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl} h_{kl} \tag{119}$$

Multipole Expansion I

- Outside the source, the solutions to Eq. (105) in the TT-gauge take the form

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{4G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{kl}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right). \tag{120}$$

- Study behavior of h_{ij}^{TT} far from the source, at a distance r that is much larger than the source's size, d.
- In that case we can expand

$$|\mathbf{x}-\mathbf{x}'| = r - \mathbf{x}'\cdot\hat{\mathbf{n}} + \mathcal{O}\left(\frac{d^2}{r}\right). \tag{121}$$

Multipole Expansion II

- To very good approximation, Eq. (120) can be written as

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{4G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|} T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c}, \mathbf{x}'\right). \tag{122}$$

- To see how further simplifications can be made, it is useful to Fourier-expand the stress tensor:

$$T_{kl}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right) = \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{kl}(\omega,\mathbf{k}) e^{-i\omega(t-r/c+\mathbf{x}'\cdot\hat{\mathbf{n}})+i\mathbf{k}\cdot\mathbf{x}'}. \tag{123}$$

- For a typical source, $T_{ij}(\omega,\mathbf{k})$ will only have power up to some maximum frequency ω_s.
- If the source is non-relativistic then $\omega_s d \ll c$.

Multipole Expansion III

- In addition we have $|\mathbf{x}'| \lesssim d$.
- Hence the frequencies ω where $h_{\mu\nu}^{\mathrm{TT}}$ receives its main contributions are such that

$$\frac{\omega}{c}\mathbf{x}' \cdot \hat{\mathbf{n}} \lesssim \frac{\omega_s d}{c} \ll 1. \tag{124}$$

- Hence, in the exponent of Eq. (123) we can use $\omega \mathbf{x}' \cdot \hat{\mathbf{n}}/c$ as an expansion parameter:

$$e^{-i\omega(t-r/c+\mathbf{x}'\cdot\hat{\mathbf{n}}/c)+i\mathbf{k}\cdot\mathbf{x}'} = e^{-i\omega(t-r/c)}\left[1 - i\frac{\omega}{c}x'^i n^i + \frac{1}{2}\left(-i\frac{\omega}{c}\right)^2 x'^i x'^j n^i n^j + \ldots\right] \tag{125}$$

Multipole Expansion IV

- In the time domain, this is equivalent to expanding

$$T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c},\mathbf{x}'\right)=T_{kl}(t-r/c,\mathbf{x}')+\frac{x'^i n^i}{c}\partial_0 T_{kl} + \frac{1}{2c^2}x'^i x'^j n^i n^j \partial_0^2 T_{kl}+\ldots, \quad (126)$$

- where the derivatives in the RHS are evaluated at $(t-r/c,\mathbf{x}')$.
- Now introduce the multipole moments of the stress tensor T_{ij}:

$$\begin{aligned} S^{ij} &= \int d^3x T^{ij}(t,\mathbf{x}), \\ S^{ij,k} &= \int d^3x T^{ij}(t,\mathbf{x})x^k, \\ S^{ij,kl} &= \int d^3x T^{ij}(t,\mathbf{x})x^k x^l, \\ &\ldots \end{aligned} \quad (127)$$

Multipole Expansion V

- Then substituting the expansion Eq. (126) into Eq. (122)

$$h_{ij}^{\mathrm{TT}} = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\left[S^{kl} + \frac{1}{c}n_m\dot{S}^{kl,m} + \frac{1}{2c^2}n_m n_p\ddot{S}^{kl,mp} + \dots\right]_{\mathrm{ret}}, \tag{128}$$

- where $[\dots]_{\mathrm{ret}}$ indicates that the expression in brackets is being evaluated at the retarded time $t - r/c$.
- Expansion in v/c, where v is a characteristic velocity.

Multipole Expansion VI

- Compared to S^{kl}, the moment $S^{kl,m}$ has an additional factor $x^m \sim \mathcal{O}(d)$
- Each time derivative brings in a factor $\mathcal{O}(\omega_s)$
- Combined with the $1/c$ this gives a factor $\mathcal{O}(\omega_s d/c)$.
- Defining $v \equiv \omega_s d$, this means that the term $(1/c)n_m \dot{S}^{kl,m}$ is a correction of $\mathcal{O}(v/c)$ to the term S^{kl}.
- Similary the term $(1/2c^2)n_m n_p \ddot{S}^{kl,mp}$ is a correction of $\mathcal{O}(v^2/c^2)$, and so on

Mass and Momenthum Multipoles I

- The expansion (128) depends on the moments of the stresses T_{ij}
- Instead have an expansion in moments of
  - mass density $(1/c^2)T^{00}$
  - momentum density $(1/c)T^{0i}$.
- The mass moments are defined as

$$
\begin{aligned}
M &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x}), \\
M^i &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})x^i, \\
M^{ij} &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})x^i x^j, \\
M^{ijk} &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})x^i x^j x^k, \\
&\ldots
\end{aligned} \tag{129}
$$

Mass and Momenthum Multipoles II

- while the momentum density moments are given by

$$
\begin{aligned}
P^i &= \frac{1}{c}\int d^3x\, T^{0i}(t,\mathbf{x}), \\
P^{ij} &= \frac{1}{c}\int d^3x\, T^{0i}(t,\mathbf{x})x^j, \\
P^{ijk} &= \frac{1}{c}\int d^3x\, T^{0i}(t,\mathbf{x})x^j x^k, \\
&\dots
\end{aligned} \tag{130}
$$

Mass and Momenthum Multipoles III

- Express the stress moments Eq. (127) as combinations of mass and momentum density moments.

$$\begin{aligned} S^{ij} &= \int d^3x\, T^{ij} \\ &= \int d^3x\, \delta^i_k \delta^j_l T^{kl} \\ &= \int d^3x\, (\partial_k x^i)(\partial_l x^j)\, T^{kl} \\ &= -\int d^3x\, x^i (\partial_l x^j)\, \partial_k T^{kl} \\ &= \int d^3x\, x^i (\partial_l x^j)\, \partial_0 T^{0l}. \end{aligned} \tag{131}$$

Mass and Momenthum Multipoles IV

- Similarly, we can write

$$\begin{aligned} S^{ij} &= -\int d^3x\, x^i x^j \partial_0^2 T^{00} - \int d^3x \delta_l^i x^j \partial_0 T^{0l} \\ &= \int d^3x\, x^i x^j \partial_0^2 T^{00} + \int d^3x^j \partial_k T^{ki} \\ &= \frac{1}{c^2} \int d^3x\, x^i x^j \ddot{T}^{00} - \int d^3x T^{ij} \\ &= \ddot{M}^{ij} - S^{ij} \qquad (132) \\ S^{ij} &= \frac{1}{2}\ddot{M}^{ij}. \qquad (133) \end{aligned}$$

Mass and Momenthum Multipoles V

- To leading order in v/c, the metric perturbation in the TT-gauge takes the form

$$\left[h_{ij}^{T}(t,\mathbf{x})\right]_{\text{quad}} = \frac{1}{r}\frac{2G}{c^2}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}^{kl}(t-r/c). \tag{134}$$

- This is the mass quadrupole radiation.
- Note that $\Lambda_{ij,kl}$ contracted with $\ddot{M}^{kl}$ makes the latter traceless,
- In Eq. (134) we can replace M^{kl} by

$$Q^{ij} \equiv M^{ij} - \frac{1}{3}\delta^{ij}M_{kk}. \tag{135}$$

- The tensor Q^{ij} related to the quadrupole tensor from Newtonian theory

$$Q^{ij} = \int d^3x\,\rho(t,\mathbf{x})\left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right). \tag{136}$$

Mass and Momenthum Multipoles VI

- In this approximation, we find

$$\left[h_{ij}^{T}(t,\mathbf{x})\right]_{\text{quad}} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{\text{TT}}(t-r/c), \tag{137}$$

- where Q_{ij}^{TT} is the transverse part of the (already traceless) tensor Q_{ij}:

$$Q_{ij}^{\text{TT}} = \Lambda_{ij,kl}(\mathbf{n})\,Q_{ij}. \tag{138}$$

Conservation of Mass and Momentum

- There is no monopole or dipole gravitational radiation.
- These contributions would have depended on time derivatives of the mass monopole M and the momentum dipole P^i.

$$\dot{M} = \frac{1}{c}\int d^3x\, \partial_0 T^{00} = -\frac{1}{c}\int d^3x\, \partial_i T^{0i} = 0, \tag{139}$$

- Can show that $\dot{P}^i = 0$.

Conservation of total mass and momentum is responsible for the absence of monopole or dipole radiation.

Quadrupole Radiation I

- Focus the quadrupole expression Eq. (137)
- What radiation is emitted depends on the direction $\hat{\mathbf{n}}$.
- However, without loss of generality we can set $\hat{\mathbf{n}} = \hat{\mathbf{z}}$,
- The projector $P_{ij} = \delta_{ij} - n_i n_j$ becomes

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{140}$$

- For any 3×3 matrix A_{ij},

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} &= \left[P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \right] A_{kl} \\ &= (PAP)_{ij} - \frac{1}{2} P_{ij} \mathrm{Tr}(PA). \end{aligned} \tag{141}$$

Quadrupole Radiation II

- Using Eq. (140) we get

$$PAP = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{142}$$

  - while $\mathrm{Tr}(PA) = A_{11} + A_{22}$.

- Therefore

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} &= \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} - \frac{A_{11} + A_{22}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \\ &= \begin{pmatrix} (A_{11} - A_{22})/2 & A_{12} & 0 \\ A_{21} & -(A_{11} - A_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} . \end{aligned} \tag{143}$$

Quadrupole Radiation III

- Thus, when $\hat{\mathbf{n}} = \hat{\mathbf{z}}$,

$$\Lambda_{ij,kl}\ddot{M}_{kl} = \begin{pmatrix} (\ddot{M}_{11} - \ddot{M}_{22})/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -(\ddot{M}_{11} - \ddot{m}_{22})/2 & 0 \\ 0 & 0 \end{pmatrix}_{ij} . \tag{144}$$

- We arrive at

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} , \tag{145}$$

Quadrupole Radiation IV

- we can immediately read off the two gravitational-wave polarizations:

$$h_+ = \frac{1}{r}\frac{G}{c^4}(\ddot{M}_{11} - \ddot{M}_{22}),$$
$$h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}, \tag{146}$$

  - where in each case the RHS is computed at the retarded time $t - r/c$.