Intensive Course in Physics
Gravitational Waves

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Chapter I: Brief Overview of General Relativity

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Mathematical Prelude

Vectors and Dual Spaces I

- Consider a position vector $\mathbf{r}(u_1, u_2, u_3)$ of $\mathcal{P}$
  - where u_1, u_2, u_3 are the coordinates of the vector in some curvilinear coordinate system (e.g. polar coordinates).
- Define the vector $\mathbf{e_1} = \frac{\partial \mathbf{r}}{\partial u_1}$ that is tangent to the u_1 curve at $\mathcal{P}$.
- In general, we can write

$$\mathbf{e_i} = \frac{\partial \mathbf{r}}{\partial u_i} \tag{1}$$

$\mathbf{e_i}$ form a basis for the curvilinear coordinate system.

Vectors and Dual Spaces II

- An infinitesimal vector displacement in a general coordinate can be written as

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3$$
$$= du_1 \mathbf{e_1} + du_2 \mathbf{e_2} + du_3 \mathbf{e_3}, \qquad (2)$$

  - where du_i are the infinitesimal displacements along the u_i curves.

Vectors and Dual Spaces III

- Consider the surface $u_1 = c_1$ where c_1 is some constant.
- The vector $\epsilon_1 = \nabla u_1$ is a vector normal to the $u_1 = c_1$ plane.
- In general, one can write

$$\epsilon_{\mathbf{i}} = \nabla u_i \tag{3}$$

These also form a set of basis vectors in this curvilinear coordinate system.

Vectors and Dual Spaces IV

- A vector $\mathbf{a}$ can therefore be written as

$$\begin{aligned}\mathbf{a} &= \alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2} + \alpha_3\mathbf{e_3} \\ &= \beta_1\epsilon_{\mathbf{1}} + \beta_2\epsilon_{\mathbf{2}} + \beta_3\epsilon_{\mathbf{3}},\end{aligned} \tag{4}$$

- $\alpha_1, \alpha_2, \alpha_3$: contravariant components of $\mathbf{a}$
- $\beta_1, \beta_2, \beta_3$: covariant components of $\mathbf{a}$

Einstein Notation I

- Useful to denote the vector $\epsilon_{\mathbf{i}}$ by $\mathbf{e^i}$.
- Position of the index (super- or subscript) distinguishes the different sets of dual vectors.
- Write the vector $\mathbf{a}$ in either basis sets as

$$\begin{aligned}\mathbf{a} &= a^1\mathbf{e_1} + a^2\mathbf{e_2} + a^3\mathbf{e_3} \\ &= a_1\mathbf{e^1} + a_2\mathbf{e^2} + a_3\mathbf{e^3},\end{aligned} \tag{5}$$

  - a^i: contravariant components of $\mathbf{a}$
  - a_i: covariant components of $\mathbf{a}$

Einstein Notation II

- Define the Einstein notation

  any index that appears exactly twice, once as a subscript and once as a superscript, in any term of an expression is understood to be summed over all the values that an index in that position can take (unless explicitly stated otherwise).

- Example: in a three-dimensional space we can write.

$$\begin{aligned} a^i b_i &= \sum_{i=1}^{3} a^i b_i \\ &= a^1 b_1 + a^2 b_2 + a^3 b_3 \end{aligned} \tag{6}$$

General Coordinate Transform I

- Consider general coordinate transformation u^i to u'^i

$$u'^i = u'^i\left(u^i\right). \tag{7}$$

- Assume that this coordinate transformation can be inverted

$$u^i = u^i\left(u'^i\right). \tag{8}$$

- The two sets of basis vectors in the new coordinate system are

$$\mathbf{e}'_i = \frac{\partial \mathbf{r}}{\partial u'^i} \text{ and } \mathbf{e}'^i = \nabla u'^i. \tag{9}$$

General Coordinate Transform II

- Use the chain rule to perform a coordinate transform

$$\begin{aligned}\mathbf{e}_i &= \frac{\partial \mathbf{r}}{\partial u^i} \\ &= \frac{\partial u'^j}{\partial u^i}\frac{\partial \mathbf{r}}{\partial u'^j} \\ &= \frac{\partial u'^j}{\partial u^i}\mathbf{e}'_j\end{aligned}$$

- Similarly, we can rewrite the second set of basis vectors as

$$\mathbf{e}^j = \frac{\partial u^j}{\partial u'^i}\mathbf{e}'^i \tag{10}$$

First Order Tensors I

- Recall from Eq. (4): write a vector $\mathbf{a}$ either in the covariant or the contravariant basis sets.
- In the contravariant form, the vector $\mathbf{a}$ written as

$$\begin{aligned}\mathbf{a} &= a'^i \mathbf{e}'_i \\ &= a^j \mathbf{e}_j \\ &= a^j \frac{\partial u'^i}{\partial u^j} \mathbf{e}'_j. \end{aligned} \tag{11}$$

- Contravariant components of a vector $\mathbf{a}$ transform as

$$a'^i = a^j \frac{\partial u'^i}{\partial u^j}. \tag{12}$$

First Order Tensors II

- Similarly, in the covariant form we can write

$$\begin{aligned} \mathbf{a} &= a'_i \mathbf{e}'^i \\ &= a_j \mathbf{e}^j \\ &= a_j \frac{\partial u^j}{\partial u'^i} \mathbf{e}'^i. \end{aligned} \tag{13}$$

- Components of a covariant vector transform as

$$a'_i = a_j \frac{\partial u^j}{\partial u'^i}. \tag{14}$$

a_i are considered the contravariant components of a first-order tensor if they transform as Eq. (12). Similarly, a_j are covariant components of a vector if it transforms as Eq. (14).

Zeroth Order Tensors

What about quantities that are unchanged by a general coordinate transformation?

- Example: length of a vector given by $r^2 = x^2 + y^2 + z^2$.
- Refer to these quantities as scalars or zeroth-order tensors.

Scalars play a crucial role in general relativity as observables

Second-order and higher-order tensors

- Generalise the discussion for first-order to tensors of higher rank.
- Example: components of a second-order tensor transform as

$$T'^{ij} = \frac{\partial u'^i}{\partial u^k} \frac{\partial u'^j}{\partial u^l} T^{kl}, \tag{15}$$

$$T'^i_j = \frac{\partial u'^i}{\partial u^k} \frac{\partial u^l}{\partial u'^j} T^k_l, \tag{16}$$

$$T'_{ij} = \frac{\partial u^k}{\partial u'^i} \frac{\partial u^l}{\partial u'^j} T_{kl}. \tag{17}$$

Metric Tensor I

Any curvilinear coordinate system is completely described at each point by a symmetric second-order tensor $\mathbf{g}$ called the metric tensor.

- Covariant and contravariant components of the metric tensor are given by

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{18}$$

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j, \tag{19}$$

- Mixed components of the metric tensor is the Kronecker delta

$$\begin{aligned} g^i_j &= \mathbf{e}^i \cdot \mathbf{e}_j \\ &= \delta^i_j, \end{aligned} \tag{20}$$

Metric Tensor II

- Suppose an infinitesimal vector displacement $d\mathbf{r} = du^i \mathbf{e}_i$.
- Write the square of the infinitesimal arc length ds^2 in terms of the metric tensor

$$\begin{aligned} ds^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= du^i \mathbf{e}_i \cdot du^j \mathbf{e}_j \\ &= g_{ij} du^i du^j. \end{aligned} \tag{21}$$

- Can also show that the volume element dV is given by

$$dV = \sqrt{g} du^1 du^2 du^3, \tag{22}$$

  - where $g = \det g_{ij}$

Metric Tensor III

- Scalar product between two vectors in terms of metric tensor.

$$\mathbf{a} \cdot \mathbf{b} = a^i \mathbf{e}_i \cdot b^j \mathbf{e}_j = a^i b^j g_{ij}, \tag{23}$$

$$\mathbf{a} \cdot \mathbf{b} = a_i \mathbf{e}^i \cdot b_j \mathbf{e}^j = a_i b_j g^{ij}, \tag{24}$$

$$\mathbf{a} \cdot \mathbf{b} = a^i \mathbf{e}_i \cdot b_j \mathbf{e}^j = a^i b^j \delta_i^j = a^i b_i \tag{25}$$

$$\mathbf{a} \cdot \mathbf{b} = a_i \mathbf{e}^i \cdot b^j \mathbf{e}_j = a_i b_j \delta_j^i = a_i b^i \tag{26}$$

- By comparing Eqs. (23)–(26)

$$g_{ij} b^j = b_i \qquad \text{and} \qquad g^{ij} b_j = b^i. \tag{27}$$

- Metric tensor can be used to raise and lower indices.
- Also works for higher-order tensors

$$T_{ij} = g_{ik} T^k_j = g_{ik} g_{jl} T^{kl}. \tag{28}$$

Derivatives of Vectors I

- In general coordinate systems, the basis vectors depend on the coordinates themselves.
- When we differentiate tensors, we must also differentiate the basis vectors.
- Consider the partial derivative $\frac{\partial \mathbf{e}_i}{\partial u^j}$: itself a vector!
- Express in terms of the basis vectors

$$\frac{\partial \mathbf{e}_i}{\partial u^j} = \Gamma^k{}_{ij} \mathbf{e}_k. \tag{29}$$

- Rearrange Eq. (29) to get

$$\Gamma^k{}_{ij} = \mathbf{e}^k \cdot \frac{\partial \mathbf{e}_i}{\partial u^j} \tag{30}$$

Derivatives of Vectors II

- Similarly, we can show that the derivative of the contravariant basis vectors are given by

$$\frac{\partial \mathbf{e}^i}{\partial u^j} = -\Gamma^i{}_{kj}\mathbf{e}^k. \tag{31}$$

- In Cartesian coordinates, basis vectors remain constant throughout the coordinate system.

Derivatives of Vectors III

- Christoffel symbol is symmetric under the interchange of the i and j subscript, because

$$\frac{\partial \mathbf{e}_j}{\partial u^j} = \frac{\partial^2 \mathbf{r}}{\partial u^j \partial u^i} = \frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j} = \frac{\partial \mathbf{e}_j}{\partial u^i}. \tag{32}$$

- Express components of Chirstoffel symbol in terms of the metric tensor

$$\Gamma^m{}_{ij} = \frac{1}{2} g^{mk} \left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right). \tag{33}$$

Covariant derivative I

In general coordinate systems, differentiation of components of a tensor with respect to the coordinates does not, in general, result in a tensor (except for zeroth-order tensors).

- Use the Christoffel symbol to introduce the covariant derivative that when acted on components of a tensor does yield components of another tensor.

Covariant derivative II

- Consider the derivative of a vector $\mathbf{v}$ in the contravariant form

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial u^j} &= \frac{\partial}{\partial u^j}\left(v^i \mathbf{e}_i\right) \\ &= \frac{\partial v^i}{\partial u^j}\mathbf{e}_i + v^i \frac{\partial \mathbf{e}_i}{\partial u^j} \\ &= \frac{\partial v^i}{\partial u^j}\mathbf{e}_i + v^i \Gamma^k{}_{ij}\mathbf{e}_k \\ &= \frac{\partial v^i}{\partial u^j}\mathbf{e}_i + v^k \Gamma^i{}_{kj}\mathbf{e}_i \\ &= \left(\frac{\partial v^i}{\partial u^j} + v^k \Gamma^i{}_{kj}\right)\mathbf{e}_i, \end{aligned} \tag{34}$$

  - where we have changed the dummy indices i and k.
- The terms in parentheses is called the covariant derivative.

Covariant derivative III

- A short-hand notation for the covariant derivative is given by

$$v^i{}_{;j} \equiv \frac{\partial v^i}{\partial u^j} + v^k \Gamma^i{}_{kj}. \tag{35}$$

- The covariant derivative of the covariant components can be shown to be

$$v_{i;j} = \frac{\partial v_i}{\partial u^j} - v^k \Gamma^k{}_{ij}. \tag{36}$$

- Introduce a similar notation for the partial derivative.

$$v^i{}_{,j} \equiv \frac{\partial v^i}{\partial u^j}. \tag{37}$$

- Follow a procedure similar to Eq. (34) to find the covariant derivative of higher order tensors.

Covariant derivative IV

- Example, the covariant derivative of a second-order tensor can be written as

$$T^{ij}{}_{;k} = T^{ij}{}_{,k} + \Gamma^i{}_{lk} T^{lj} + \Gamma^j{}_{lk} T^{il}, \quad (38)$$

$$T^i{}_{j;k} = T^i{}_{j,k} + \Gamma^i{}_{lk} T^l_j - \Gamma^l{}_{jk} T^i_l, \quad (39)$$

$$T_{ij;k} = T_{ij,k} - \Gamma^l{}_{ik} T_{lj} - \Gamma^l{}_{jk} T_{il}. \quad (40)$$

- Higher order tensors: For each contravariant index use a Christoffel symbol with a plus sign, and for a covariant index we use a Christoffel symbol with a minus sign.

Absolute derivative I

- Consider a derivative of a vector $\mathbf{v}(\mathbf{t})$ along some curve parametrised by t.

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{dv^i}{dt}\mathbf{e}_i + v^i \frac{d\mathbf{e}_i}{dt} \\
&= \frac{dv^i}{dt}\mathbf{e}_i + v^i \frac{\partial \mathbf{e}_i}{\partial u^k}\frac{du^k}{dt} \\
&= \frac{dv^i}{dt}\mathbf{e}_i + \Gamma^j{}_{ik} v^i \frac{du^k}{dt}\mathbf{e}_j \\
&= \left(\frac{dv^j}{dt} + \Gamma^i{}_{jk}\frac{du^k}{dt}\right)\mathbf{e}_j \\
&= \left(v^j{}_{;k}\frac{du^k}{dt}\right)\mathbf{e}_j, \qquad (41)
\end{aligned}
$$

Absolute derivative II

- The term inside the parenthesis is called the absolute derivative

$$\frac{\delta v^i}{\delta t} \equiv v^i{}_{;k} \frac{du^k}{dt}, \tag{42}$$

- For covariant components, we have

$$\frac{\delta v_i}{\delta t} \equiv v_{i;k} \frac{du^k}{dt}. \tag{43}$$

Absolute derivative III

- For second-order tensors, we can arrive at similar expressions

$$\frac{\delta T^{ij}}{\delta t} \equiv T^{ij}{}_{;k}\frac{du^k}{dt}, \tag{44}$$

$$\frac{\delta T^{i}{}_{j}}{\delta t} \equiv T^{i}{}_{j;k}\frac{du^k}{dt}, \tag{45}$$

$$\frac{\delta T_{ij}}{\delta t} \equiv T_{ij;k}\frac{du^k}{dt}. \tag{46}$$

- These expression can be generalised to tensors of arbitrary orders.

Geodesics I

A geodesic is a generalization of the notion of a straight line to curved spaces and has two equivalent properties

1. the curve of the shortest length between two points,
2. the curve whose tangent vectors remain parallel when transported along the curve.

Geodesics II

- Consider a curve $\mathbf{r}(s)$, which is parameterised by the arc length s starting from some point on the curve.
- The tangent vector is given by $\mathbf{t} = d\mathbf{r}/ds$.
- Find the geodesic by the property that the tangent vector remains parallel moving along the curve,

$$\frac{d\mathbf{t}}{ds} = 0. \tag{47}$$

Geodesics III

- Tangent vector $\mathbf{t}$ is given by

$$\mathbf{t} = t^i \mathbf{e}_i \tag{48}$$

- The geodesic can be found by evaluating the absolute derivative

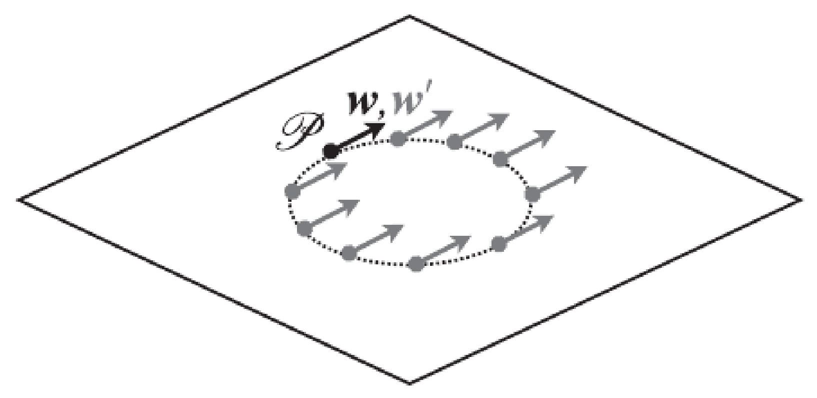
$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= t^i{}_{;k} \frac{du^k}{ds} \mathbf{e}_i \\ &= \left(\frac{dt^i}{ds} + \Gamma^i{}_{jk} t^j \right) \frac{du^k}{ds} \mathbf{e}_i \\ &= 0. \end{aligned} \tag{49}$$

- Since $t^j = du^j/ds$, we can find an alternative expression for the geodesic

$$\frac{d^2u^i}{ds^2} + \Gamma^i{}_{jk} \frac{du^j}{ds} \frac{du^k}{ds} = 0 \tag{50}$$

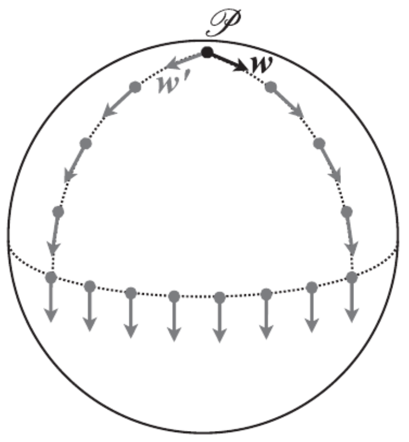
Parallel Transport I

Consider parallel transporting a vector along a closed loop.



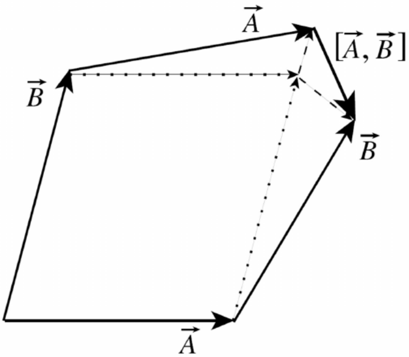
Parallel Transport II

Measure the intrinsic curvature by parallel transportation along a closed loop



Riemann Curvature Tensor I

- Recall: parallel transport along u^k is given by $t^i{}_{;k}$.
- Go around an infinitesimal square in the u^j and u^k direction.



Riemann Curvature Tensor II

- Change in a vector **v** given by commutator of two covariant derivatives

$$v_{i;[j,k]} = v_{i;jk} - v_{i;kj} \equiv R^l{}_{ijk} v_l, \tag{51}$$

  - where $R^l{}_{ijk}$ is the so-called Riemann tensor.
- The Riemann tensor provides a measure of the curvature.

Riemann Curvature Tensor III

- Can be rewritten as

$$R^{\beta}{}_{\alpha\mu\nu} = \left(\Gamma^{\beta}{}_{\alpha\nu,\mu} - \Gamma^{\beta}{}_{\alpha\mu,\nu} + \Gamma^{\gamma}{}_{\alpha\nu}\Gamma^{\beta}{}_{\gamma\mu} - \Gamma^{\gamma}{}_{\alpha\mu}\Gamma^{\beta}{}_{\gamma\nu}\right). \quad (52)$$

- Riemann tensor has the following properties

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= -R_{\mu\nu\beta\alpha}, \\ R_{\mu\nu\alpha\beta} &= -R_{\nu\mu\alpha\beta}, \\ R_{\mu\nu\alpha\beta} &= -R_{\alpha\beta\mu\nu}, \end{aligned} \quad (53)$$

- Satisfies the Bianchi identities

$$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} = 0. \quad (54)$$

Riemann Curvature Tensor IV

- Define the Ricci curvature tensor as the contraction of the Riemann tensor

$$R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta}, \tag{55}$$

- Ricci scalar/curvature as the contraction of the Ricci curvature tensor.

$$R \equiv R^{\alpha}{}_{\alpha} \tag{56}$$

- Define the divergence-free Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{57}$$

Stress-Energy Tensor I

In general relativity, the single Newtonian potential Φ is replaced with ten potentials $g_{\mu\nu}$

- Describe the source of gravity as a Stress-energy tensor
  - Energy density: ρ
  - Energy flux: $\mathbf{j} = \rho\mathbf{v}$
  - Stress tensor: $dF_i = S_{ij}\hat{n}^j dA$

$$T_{ij} = \begin{pmatrix} \rho & j_j \\ j_i & S_{ij} \end{pmatrix} \tag{58}$$

- Conservation of energy states

$$\nabla_\mu T^{\mu\nu} = 0 \tag{59}$$

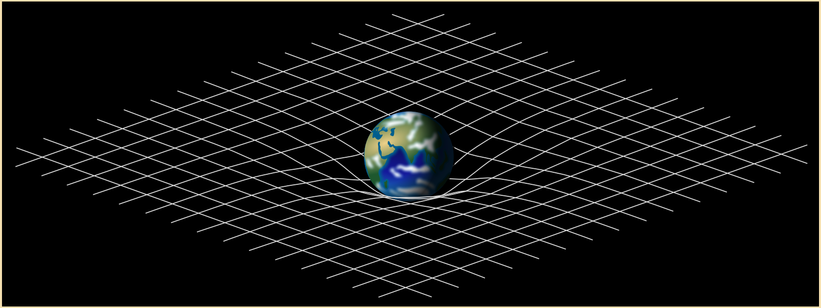
General Relativity

General theory of relativity

Spacetime tells matter how to move; matter tells spacetime how to curve.

John A. Wheeler

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$



Pseudo-Riemannian manifolds

Spacetime is a manifold that is continuous and differentiable.

- Define scalars, vectors, 1-forms and general tensor fields
- Able to take derivatives at any point
- Locally, these points are ordered as points in a Euclidian space
- We specify a distance concept by adding a metric $\mathbf{g}$, which contains information about how fast clocks proceed and what are the distances between points.
- A differentiable manifold with a metric as additional structure, is termed a (pseudo-)Riemannian manifold.

Local Lorentz Frame I

- We now want to assign a metric to spacetime.
- Introduce a local Lorentz frame (LLF).
  - Freefall at point $\mathcal{P}$.
  - The equivalence principle: all effects of gravitation disappear and that we locally obtain the metric of the special theory of relativity
  - This is the Minkowski metric

Local Lorentz Frame II

- While in special relativity this can be a *global* coordinate system, in general relativity (GR) this is only *locally* possible.
- The metric becomes $g_{\mu\nu} \to \eta_{\mu\nu}$

$$\eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1) \tag{60}$$

- Define distances using $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$
- For a Riemannian manifold all diagonal elements need to be positive.
- The signature (the sum of the diagonal elements) of the metric of spacetime is $+2 \to$ pseudo-Riemannian.

Curved spacetime I

- In a curved spacetime we cannot define a global Lorentz frame for which $g_{\alpha\beta} = \eta_{\alpha\beta}$.
- However, it is possible to choose coordinates such that in the vicinity of $\mathcal{P}$ this equation is *almost* valid.
    - Equivalence principle.

Curved spacetime II

- For such a coordinate system one has

$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta} \tag{61}$$

$$\frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(\mathcal{P}) = 0 \tag{62}$$

$$\frac{\partial^2}{\partial x^\gamma \partial x^\mu} g_{\alpha\beta}(\mathcal{P}) \neq 0 \tag{63}$$

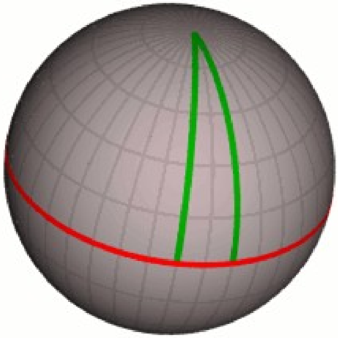
- The existence of local Lorentz frames expresses that each curved spacetime has at each point a flat tangent space.
- All tensor manipulations occur in this tangent space.

Newtonian Tidal forces I

How to find a measure of the curvature of spacetime?

- Drop a single test particle?
  - Go along in free-fall
  - Particle at rest (straight line in time direction)
  - Nothing that betrays curvature
  - A single particle is insufficient to discover effects of curvature.

Newtonian Tidal forces II



Newtonian Tidal forces III

- Drop two test particles?
  - Free-fall observers fall in straight line towards center of Earth
  - Both particles follow paths that lead to the center of the Earth
  - Particles move towards each other $\rightarrow$ Tidal forces
  - According to Newton both paths interact because of gravitation,
  - According to Einstein this occurs because spacetime is curved.

Gravitation is a property of the curvature of spacetime

Newtonian Tidal forces IV

- The Newtonian equations of motion for particles P and Q are

$$\left(\frac{d^2 x_j}{dt^2}\right)_{(P)} = -\left(\frac{\partial \Phi}{\partial x^j}\right)_{(P)} \tag{64}$$

$$\left(\frac{d^2 x_j}{dt^2}\right)_{(Q)} = -\left(\frac{\partial \Phi}{\partial x^j}\right)_{(Q)}, \tag{65}$$

  with Φ being the gravitational potential.

- Define $\vec{\xi} = (x_j)_{(P)} - (x_j)_{(Q)}$ as the separation between both particles.
- For parallel trajectories one has $\frac{d\vec{\xi}}{dt} = 0$.

Newtonian Tidal forces V

- Taylor expansion to leading order in $\vec{\xi}$ gives

$$\frac{d^2\xi_j}{dt^2} = -\left(\frac{\partial^2\Phi}{\partial x^j \partial x^k}\right)\xi_k \tag{66}$$

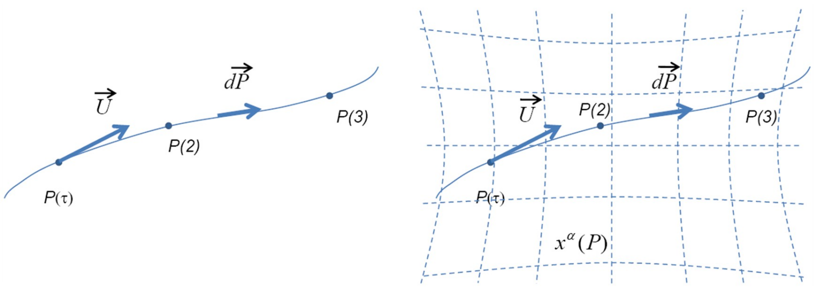
$$= -\mathcal{E}_{jk}\xi_k \tag{67}$$

- And we define the gravitational tidal tensor $\mathcal{E}$

$$\mathcal{E}_{jk} = \left(\frac{\partial^2\Phi}{\partial x^j \partial x^k}\right), \tag{68}$$

Newtonian geodesic deviation.

Einstein equations I



- Consider a particle along a worldline.
- This worldline is parameterized with proper time τ on a clock that is carried by the particle.
- Denote the position of the particle at a point of the worldline with $\mathcal{P}(\tau)$

Einstein equations II

- The velocity $\vec{U}$ is the tangent vector of the curve and is given by

$$\vec{U} = \frac{d\mathcal{P}}{d\tau} = \frac{d}{d\tau}. \tag{69}$$

- For the velocity in the LLF at point $\mathcal{P}$

$$\vec{U}^2 = \frac{\overrightarrow{d\mathcal{P}} \cdot \overrightarrow{d\mathcal{P}}}{d\tau^2} \tag{70}$$

$$= \frac{-d\tau^2}{d\tau^2} \tag{71}$$

$$= -1, \tag{72}$$

- Because this equation yields a number (scalar), it is valid in every coordinate system.

Einstein equations III

- Four-velocity vector has length 1 and points in the time direction.
- Components of the velocity are given by

$$U^\alpha = \frac{dx^\alpha}{d\tau}. \tag{73}$$

Einstein equations IV

- Consider a particle moving freely
- Must move in a straight line (parallel transport its own velocity)

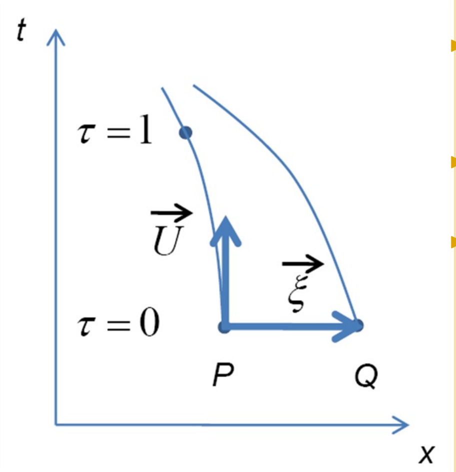
$$\nabla_{\vec{U}}\vec{U} = 0, \tag{74}$$

or

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\ \mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{75}$$

- which is the expression for a geodesic.

Einstein equations V



- Suppose we have two particles that at a certain instant ($\tau = 0$)
- At rest with respect to each other.
- We define the separation vector $\vec{\xi}$, which points from one particle to the other.

$$\nabla_{\vec{U}}\vec{\xi} = 0 \qquad (76)$$

Einstein equations VI

- Demand that the particles are initially ($\tau = 0$) at rest with respect to each other
- Define $\vec{\xi}$ such that in the LLF of particle P this vector $\vec{\xi}$ is purely spatial

$$\left.\begin{array}{rcl} \nabla_{\vec{U}}\vec{\xi} & = & 0 \\ \vec{U}\cdot\vec{\xi} & = & 0 \end{array}\right\} \quad \text{at point } \mathcal{P} \quad \text{for } \tau = 0. \tag{77}$$

- The second derivative $\nabla_{\vec{U}}\nabla_{\vec{U}}\vec{\xi}$ does not vanish.

Geodesics of the particles are forced together or apart (depending on the metric) when time progresses.

Einstein equations VII

- One can now write

$$\nabla_{\vec{U}}\nabla_{\vec{U}}\vec{\xi} = -\mathbf{R}(\_, \vec{U}, \vec{\xi}, \vec{U}), \tag{78}$$

with $\mathbf{R}$ being the curvature tensor.

- In the LLF of particle P at time $\tau = 0$ one has $U^0 = 1$ and $U^i = 0$.

$$(\nabla_{\vec{U}}\nabla_{\vec{U}}\vec{\xi})^j = \frac{\partial^2 \vec{\xi}^j}{\partial t^2} = -R^j_{\alpha\beta\gamma}U^\alpha \xi^\beta U^\gamma = -R^j_{0k0}\xi^k, \tag{79}$$

  - since the velocity $\vec{U}$ only has a non-vanishing time component in the LLF of particle $\mathcal{P}$
  - while the separation vector $\vec{\xi}$ only has spacelike components $k = 1, 2, 3$.

Einstein equations VIII

- In the LLF the geodesic deviation is given by

$$\frac{\partial^2 \xi^j}{\partial t^2} = -R^j_{0k0}\xi^k, \tag{80}$$

- while in Newtonian mechanics we have found that

$$\frac{\partial^2 \xi^j}{\partial t^2} = -\mathcal{E}_{jk}\xi^k. \tag{81}$$

- Comparing both expressions yields

$$R_{j0k0} = \mathcal{E}_{jk} = \frac{\partial^2 \Phi}{\partial x^j x^k}. \tag{82}$$

Einstein equations IX

- According to Newton one has

$$\nabla^2\Phi = 4\pi G\rho \quad \rightarrow \quad \partial_j\partial_k\Phi\,\delta^{jk} = \mathcal{E}_{jk}\delta^{jk} = \mathcal{E}^j{}_j, \tag{83}$$

- we find for the trace of the gravitational tidal tensor

$$\mathcal{E}^j{}_j = 4\pi G\rho \tag{84}$$

- In analogy one might expect that in GR one has

$$R^j{}_{0j0} = 4\pi G\rho \quad ? \tag{85}$$

Einstein equations X

- Should not depend on the choice of coordinate system!
  - The equation exist in a special system: the LLF.
- Find a relation between tensors.
  - In the LLF one has $R_{0000} = 0$ en ${R^0}_{000} = 0$ because of antisymmetry.

$$ {R^j}_{0j0} = 4\pi G\rho \to {R^\mu}_{0\mu 0} = 4\pi G\rho \tag{86} $$

Einstein equations XI

- Another difficulty: at the left of the equal sign we have two indices, while at the right there are none.
- Thus, one might expect that

$$R_{\alpha\beta} = 4\pi G T_{\alpha\beta} \quad ? \tag{87}$$

  - where $T_{\alpha\beta}$ represents the energy stress tensor, with $T_{00} = \rho$.

Einstein made this guess already in 1912, but it is incorrect!

Einstein equations XII

- We can show that the Ricci tensor is given by

$$R_{\alpha\gamma} \approx \partial^\beta \partial_\gamma g_{\alpha\beta} + \text{non-linear terms.} \tag{88}$$

- Proposed equations constitute a set of 10 partial differential equations for the 10 components of the metric $g_{\alpha\beta}$
- But we are at liberty to choose the coordinate system where we are going to work.
  - Set 4 of the 10 components of $g_{\alpha\beta}$
- However, we would have 10 partial differential equations for 6 unknowns.

What we need are 6 equations for 6 unknowns

Einstein equations XIII

- We can also consider the conservation laws for energy and momentum.

$$\nabla_\beta T^{\alpha\beta} = 0. \tag{89}$$

- But the LHS does not obey this divergence criterion

$$\nabla_\beta R^{\alpha\beta} \neq 0. \tag{90}$$

- Instead, it follows from the Bianchi identities that

$$\nabla_\beta G^{\alpha\beta} = 0 \tag{91}$$

- where $G_{\alpha\beta}$ is defined as

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}, \tag{92}$$

Einstein equations XIV

- It seems reasonable to assume that Nature has chosen

$$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}. \tag{93}$$

- Which are exactly the Einstein equations.
- The proportionality factor $(8\pi G/c^4)$ can be found by taking the Newtonian limit.