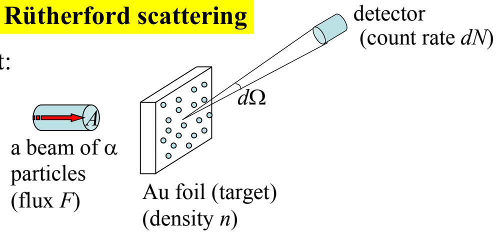
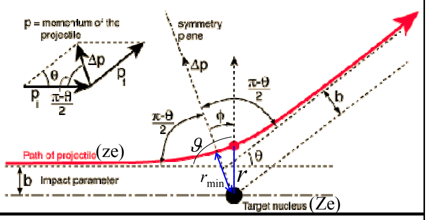
Appendix 13: Rütherford scattering

The experiment:



The α particles actually see nuclei and electron cloud in foil but they are scattered only by the nuclei.

Elastic scattering: energy of α particle: $T_a = \frac{1}{2}mv_0^2$



Assume:
(1) only Coulomb force,
(2) the electron cloud can be neglected,
(3) the target nucleus is stationary at all time.

(A) Trajectory of the scattered particle:

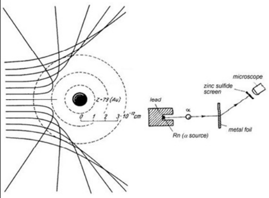
$$\frac{1}{r} = \frac{1}{b}\sin\vartheta + \frac{d_{\min}}{2b^2}(\cos\vartheta - 1) \qquad [1]$$

Here $\vartheta = \dfrac{180° - \theta}{2} + \phi$ (See last graph)

$d_{\min}$ is defined in Eq. [3].

Derivation can be found in, e.g., Eisberg, Fundamentals of Modern Physics (QC173.E358) or Eisberg & Resnick, Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles, (QC174.12.E34).

(B) Scattering angle θ is larger for smaller b:



$$\tan\frac{\theta}{2} = \frac{d_{\min}}{2b} \qquad [2]$$

where $$T_a = \frac{zZe^2}{4\pi\varepsilon_0 d_{\min}} \qquad [3]$$

(derived in Ch. 4)

(C) Distance of closest approach ($r_{\min}$)

By conservation of energy,

$$\frac{1}{2}mv^2 + \frac{zZe^2}{4\pi\varepsilon_0 r} = \text{constant} = \frac{1}{2}mv_0^2 \equiv T_a$$

At $r_{\min}$, $$\frac{1}{2}mv_{\min}^2 + \frac{zZe^2}{4\pi\varepsilon_0 r_{\min}} = T_a \qquad [4]$$

By conservation of angular momentum (⇐ central force),

$$mv_0 b = mv_{\min} r_{\min} \qquad [5]$$

Eqs. [4] & [5] ⇒ $$\frac{b^2}{r_{\min}^2} + \frac{d_{\min}}{r_{\min}} - 1 = 0 \qquad [6]$$

For $b = 0$, $\theta = 180°$, $r_{\min} = d_{\min}$

Eqs. [2] & [6] ⇒ $$r_{\min} = \frac{b\cos\frac{\theta}{2}}{1 - \sin\frac{\theta}{2}} \qquad [7]$$

The same result can be obtained from Eq. [1].

(D) Cross-section

All scatterings with impact parameter < b have scattering angle > θ.
Consider a disk of radius b.

The disk area is (by Eq. [2]) $\pi b^2 = \pi\left(\dfrac{d_{\min}}{2}\cot\dfrac{\theta}{2}\right)^2$

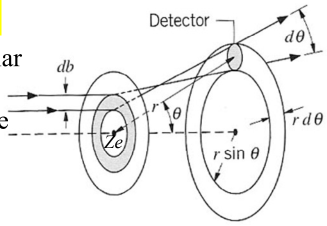
Write this area as $$\sigma(\theta) = \frac{\pi d_{\min}^2}{4}\cot^2\frac{\theta}{2} \qquad [8]$$

This is the "cross-section" for scattering through an angle > θ or impact parameter < b.
Assume that the target foil thickness D is small so that there is no multiple scattering and the α particles in the beam (flux F) hit the target with random impact parameter.
Then the scattering rate for scattering angle larger than θ is

$$N = F[n(AD)]\sigma$$

(E) Differential cross-section

Impact factors within the annular ring (width db) will lead to scattering within the solid angle (subtended at the nucleus):



$$d\Omega = 2\pi\sin\theta\, d\theta$$

Eq. [8] ⇒ $$d\sigma = -\frac{\pi d_{\min}^2}{4}\cot\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}\,d\theta$$

(Note: $db > 0 \Rightarrow d\theta < 0$)

$$\therefore \frac{d\sigma}{d\Omega} = \frac{d_{\min}^2}{16}\operatorname{cosec}^4\frac{\theta}{2}$$

or $$\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2}{16\pi\varepsilon_0 T_a}\right)^2\operatorname{cosec}^4\frac{\theta}{2} \qquad [9]$$

Then the scattering rate per unit solid angle at scattering angle θ is

$$\frac{dN}{d\Omega} = F[n(AD)]\frac{d\sigma}{d\Omega}$$

(F) The effect of electrons

By Eq. [7], $r_{\min} \sim b$ if b is not large.

The α particle is scattered significantly only when it is very close to the nucleus.

On the other hand,

Eq. [9] ⇒ $\lim_{\theta\to 0°}\dfrac{d\sigma}{d\Omega} = \infty$

This is just because for $\theta \to 0$, $b \to \infty$.
Electron screening cannot be neglected very small θ.

(G) Recoil of target nucleus

The target nucleus is subject to recoil due to conservation of linear momentum. The recoil energy depends on θ. It is maximum for $\theta = 180°$ (backward scattering).
The differential cross-section equation derived before can still be used if we consider the scattering in the CM frame and the scattering center is now the CM.

(H) Scattering of other charge particles

The same differential cross-section equation can be used for any scattering process that involves only the Coulomb force, e.g, ep scattering, eq scattering in Ch. 4. But you need proper correction for the relativistic effect.

The following note was prepared by a TA

Differential cross section for Rutherford scattering

Rutherford scattering : Elastic electric scattering of a beam of charged particles by the nucleus

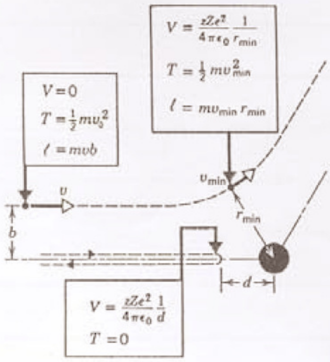


Figure 1 The trajectory of a particle undergoing Rutherford scattering, showing the closest approach to the target nucleus.

V : potential energy
T : kinetic energy
ℓ : angular momentum
b : impact parameter
d : distance of closest approach

Consider a particle of incident K.E. $T_0 = \frac{1}{2}mv_0^2$ approaching the target nucleus. Conservation of energy gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{4\pi\epsilon_0}\frac{zZe^2}{r} \qquad (1)$$

For a head-on collision $(b=0)$, we have

$$\frac{1}{2}mv_0^2 = \frac{1}{4\pi\epsilon_0}\frac{zZe^2}{d} \qquad (2)$$

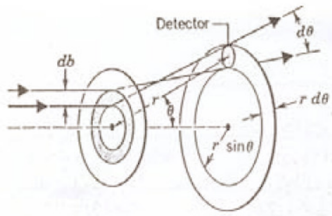


Figure 2 Particles entering the ring between b and $b + db$ are distributed uniformly along a ring of angular width $d\theta$. A detector is at a distance r from the scattering foil.

θ : scattering angle
n : target nuclei per unit volume
x : thickness of the foil

The scattering has cylindrical symmetry about the beam axis.
Assume the foil to be thin enough to ignore 'shadowing'.
The fraction of the incident particles that pass through the annular ring of area $2\pi b\,db$ is

$$df = nx(2\pi b\,db) \qquad (3)$$

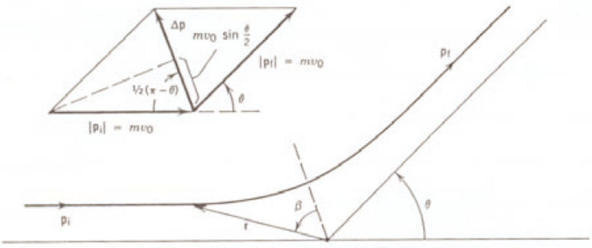


Figure 3 The hyperbolic trajectory of a scattered particle. The instantaneous coordinates are r, β. The change in momentum is Δp, in the direction of the dashed line that bisects $(\pi - \theta)$.

From Figure 3, $\Delta p = 2mv_0 \sin\frac{\theta}{2}$ in the direction of the bisector of $\pi - \theta$. (4)

$$\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\beta}{dt}\hat{\beta} \qquad (5)$$

$$\ell = |m\vec{r}\times\vec{v}| = mr^2\frac{d\beta}{dt} \qquad (6)$$

Conservation of angular momentum gives

$$mv_0 b = mr^2\frac{d\beta}{dt}$$

$$\frac{dt}{r^2} = \frac{d\beta}{v_0 b}$$

According to Newton's second law in the form of dp/dt,

$$\Delta p = \int F\,dt = \frac{zZe^2}{4\pi\epsilon_0}\int\frac{dt}{r^2}\cos\beta$$

$$= \frac{zZe^2}{4\pi\epsilon_0 v_0 b}\int_{-(\pi/2-\theta/2)}^{+(\pi/2-\theta/2)}\cos\beta\,d\beta$$

$$= \frac{zZe^2}{2\pi\epsilon_0 v_0 b}\cos\frac{\theta}{2} \qquad (7)$$

Equating (4) and (7) with the use of (2), we have

$$b = \frac{d}{2}\cot\frac{\theta}{2} \qquad (8)$$

$$|db| = \frac{d}{4}\csc^2\frac{\theta}{2}\,d\theta \qquad (9)$$

$$\therefore\ |df| = \pi n x\frac{d^2}{4}\cot\frac{\theta}{2}\csc^2\frac{\theta}{2}\,d\theta \qquad (10)$$

A differential cross section is defined by the probability to observe a scattered particle per unit solid angle, which is

$$\frac{d\sigma}{d\Omega} = \frac{\text{Scattered flux / Unit of solid angle}}{\text{Incident flux / Unit of surface}}$$

$$= \frac{1}{nx}\frac{|df|}{d\Omega}$$

$$= \left(\frac{zZe^2}{4\pi\epsilon_0}\right)^2\left(\frac{1}{4T_0}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}$$

$$\propto \frac{Z^2}{T_0^2\sin^4\frac{\theta}{2}}$$