

Preserving qubit coherence by dynamical decoupling

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In quantum information processing, it is vital to protect the coherence of qubits in noisy environments. Dynamical decoupling (DD), which applies a sequence of flips on qubits and averages the qubit-environment coupling to zero, is a promising strategy compatible with other desired functionalities, such as quantum gates. Here, we review the recent progresses in theories of dynamical decoupling and experimental demonstrations. We give both semiclassical and quantum descriptions of the qubit decoherence due to coupling to noisy environments. Based on the quantum picture, a geometrical interpretation of DD is presented. The periodic Carr–Purcell–Meiboom–Gill DD and the concatenated DD are reviewed, followed by a detailed exploration of the recently developed Uhrig DD, which employs the least number of pulses in an unequally spaced sequence to suppress the qubit–environment coupling to a given order of the evolution time. Some new developments and perspectives are also discussed.

Keywords qubit, decoherence, dynamical decoupling

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1 Coherence and decoherence

The power of quantum information processing [1], the quantum parallelism, comes from the superposition principle of quantum mechanics. The building block of quantum technology, a quantum bit (qubit), is a two-level system that can be identified as a spin-1/2 with states $|\uparrow\rangle$ and $|\downarrow\rangle$. The ability of the qubit to be in a coherent superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$,

$$|\Psi\rangle = \cos\frac{\theta}{2}e^{-i\varphi/2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\varphi/2}|\downarrow\rangle \quad (1)$$

enables the parallel processing of many pieces of classical information. In order for this idea to work, the qubit has to faithfully maintain its quantum state. Not only the populations $\cos^2(\theta/2)$ and $\sin^2(\theta/2)$ in the two states $|\uparrow\rangle$ and $|\downarrow\rangle$, but also the relative phase $e^{-i\varphi}$ between $|\uparrow\rangle$ and $|\downarrow\rangle$ should be kept at certain values. Unavoidable couplings between the qubit and the environment (hereafter referred to as bath) spoil the quantum state by introducing uncontrolled evolution of the qubit. The populations

and phases are randomized, and the qubit coherence is lost. This decoherence problem is one of the most serious obstacles in the roads toward scalable quantum information processing [2].

The population randomization (i.e., relaxation) process involves energy dissipation and therefore is subjected to the energy conservation condition. Thus, it can be suppressed by increasing the spin splitting of the qubit. In contrast, the phase randomization (i.e., pure dephasing) is a more serious issue, since this process does not involve energy dissipation.

1.1 Semiclassical picture of decoherence

In the semiclassical picture, the pure dephasing of a qubit or a spin-1/2 is caused by the fluctuation of a local classical field fixed at a given direction [3, 4]. The Hamiltonian of the qubit in the external field including the random component is

$$\hat{H}_{\text{qubit}} = \left[\frac{\omega_0}{2} + Z(t) \right] \hat{\sigma}_z \quad (2)$$

where $\hat{\sigma}_z$ is the Pauli matrix for the qubit, ω_0 is the Zeeman splitting under the external field, and $2Z(t)$ is the random field resulting from the interaction with the bath. Let us consider a qubit initially in a coherent superposition state

$$|\psi(0)\rangle = C_+ |\uparrow\rangle + C_- |\downarrow\rangle \quad (3)$$

corresponding to a pure state density matrix

$$\hat{\rho}(0) = \begin{pmatrix} |C_+|^2 & C_+ C_-^* \\ C_+^* C_- & |C_-|^2 \end{pmatrix} \quad (4)$$

in the basis $|\uparrow\rangle, |\downarrow\rangle$. At the end of the evolution, a random relative phase $\varphi(\tau) = 2 \int_0^\tau Z(t) dt$ between $|\uparrow\rangle$ and $|\downarrow\rangle$ is accumulated in the qubit wave function

$$|\psi(\tau)\rangle = C_+ e^{-i\varphi(\tau)/2} |\uparrow\rangle + C_- e^{i\varphi(\tau)/2} |\downarrow\rangle \quad (5)$$

and the off-diagonal coherence of the resulting density matrix

$$\hat{\rho}(\tau) = \begin{pmatrix} |C_+|^2 & C_+ C_-^* e^{-i\varphi(\tau)} \\ C_+^* C_- e^{i\varphi(\tau)} & |C_-|^2 \end{pmatrix} \quad (6)$$

becomes random. The ensemble average over all possible realizations of the random noise $Z(t)$ gives the decay of the off-diagonal density matrix elements, i.e., the decoherence of the qubit (or the depolarization of the spin-1/2 in the plane perpendicular to the external field). The resulting qubit state is a mixed state with vanishing off-diagonal coherence, since the noise-averaged quantity $\langle e^{-i\varphi(\tau)} \rangle$ vanishes in the long time limit.

1.2 Quantum theory of decoherence

In the quantum picture [5], the decoherence of a qubit results from the qubit–bath entanglement, which is established during the evolution of the interacting qubit–bath system. The general pure dephasing Hamiltonian has the form

$$\hat{H}_{\text{pd}} = \hat{C} + \hat{\sigma}_z \otimes \hat{Z} \quad (7)$$

where \hat{C} is the interaction within the bath, and \hat{Z} is the bath operator representing the quantum field on the qubit resulting from the qubit–bath interaction. Suppose the initial state of the qubit–bath system has the form $|\Psi(0)\rangle = |\psi(0)\rangle \otimes |J\rangle$, i.e., a direct product of the qubit state $|\psi(0)\rangle = C_+ |\uparrow\rangle + C_- |\downarrow\rangle$ and the bath state $|J\rangle$. At the end of the evolution, an entangled state is established as

$$\begin{aligned} |\Psi(\tau)\rangle &= C_+ |\uparrow\rangle \otimes e^{-i(\hat{C} + \hat{Z})\tau} |J\rangle \\ &\quad + C_- |\downarrow\rangle \otimes e^{-i(\hat{C} - \hat{Z})\tau} |J\rangle \\ &\equiv C_+ |+\rangle \otimes |J^{(+)}(\tau)\rangle + C_- |-\rangle \otimes |J^{(-)}(\tau)\rangle \end{aligned} \quad (8)$$

and the off-diagonal coherence of the reduced density matrix of the qubit becomes bath-state dependent:

$$\hat{\rho}(\tau) = \begin{pmatrix} |C_+|^2 & C_+ C_-^* \langle J^{(-)}(\tau) | J^{(+)}(\tau) \rangle \\ C_+^* C_- \langle J^{(+)}(\tau) | J^{(-)}(\tau) \rangle & |C_-|^2 \end{pmatrix} \quad (9)$$

The off-diagonal qubit coherence is reduced when the bath state overlap decreases

$$\begin{aligned} L_J(\tau) &\equiv \langle J^{(-)}(\tau) | J^{(+)}(\tau) \rangle \\ &= \langle J | e^{i(\hat{C} - \hat{Z})\tau} e^{-i(\hat{C} + \hat{Z})\tau} | J \rangle \end{aligned} \quad (10)$$

A transparent physical meaning of this formula is that the coherence of the qubit decreases when the distinguishability of the bath states increases, or the quantumness of the qubit decays when it is gradually “measured” by the environment.

The decoherence in Eq. (10) is caused by the quan-

tum fluctuation of the local field for a single bath state $|J\rangle$. At finite temperature, the bath itself is in a thermal ensemble as $\sum_J P_J |J\rangle \langle J|$. Ensemble average over the distribution of the initial bath states $|J\rangle$ causes additional dephasing due to the thermal fluctuation, which is referred to as inhomogeneous broadening in literature [6].

As an example, in a confined solid-state environment, such as a quantum dot, the most relevant source of decoherence at low temperature (a few Kelvin) for an electron spin is the hyperfine interaction with the lattice nu-

clear spins (which serve as the bath) [6–9]. In a moderate ($\gtrsim 0.1$ Tesla in GaAs quantum dots) external magnetic field, the electron spin relaxation is strongly suppressed [10–12], and the coherence decay is dominated by pure dephasing. Recently, a variety of quantum many-body theories have been developed to evaluate the bath state evolution $L_J(\tau)$ or its ensemble average, including the density matrix cluster expansion [13, 14], the pair-correlation approximation [5], the linked-cluster expansion [15], and the cluster correlation expansion [16, 17]. In the pair-correlation approximation [5], each pair-wise flip-flop of the nuclear spins is identified as an elementary excitation mode and is taken as independent of each other. To study the higher order correlations, the Feynman diagram linked-cluster expansion is developed [15]. The evaluation of higher order linked-cluster expansion, however, is tedious due to the increasing number and complexity of diagrams with increasing the interaction order. The density matrix cluster expansion [13, 14] provides a simple solution to include the higher order spin interaction effects beyond the pair-correlation approximation (without the need to count or evaluate Feynman diagrams). However, the accuracy problem (even when the expansion converges) limits the cluster expansion to applications in large spin baths. The cluster-correlation expansion [16, 17] bears the accuracy of the linked-cluster expansion (the results are accurate whenever converge) and the simplicity of the cluster expansion (without the need to count or evaluate Feynman diagrams), while free from the large-bath restriction of the cluster expansion.

2 Suppressing decoherence by dynamical decoupling

Since qubit decoherence results from uncontrolled evolution due to the coupling between the qubit and the bath, a natural idea to combat decoherence is to encode the qubit in a subspace immune to noises from the environment (decoherence-free subspace [18, 19]), which is made possible by symmetries of the interactions in certain physical systems. Alternatively, the coherence can be protected by dynamically eliminating the qubit-bath coupling during the evolution (dynamical decoupling, referred to as DD for short). The DD schemes were originated from the Hahn echo [20] and were developed for high-precision magnetic resonance spectroscopy [21–23]. When the field of quantum computing was opened up, the idea of DD was introduced to protect qubit coherence [24–27], which stimulated numerous theoretical studies on extension and applications of the DD approach to quantum computing [28–38]. The recent experimental advances are also remarkable [39–41].

In the DD scheme, a sequence of pulses is applied to

flip the qubit and average the qubit–bath coupling to zero during the evolution. It is a promising strategy due to its compatibility with other desired functionalities, such as quantum gates [42–44]. The most general Hamiltonian describing the coupling between a qubit and a bath reads

$$\hat{H} = \hat{C} + \hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y} + \hat{\sigma}_z \otimes \hat{Z} \quad (11)$$

where $\hat{\sigma}_{x/y/z}$ are the Pauli matrices for the qubit, and \hat{C} , \hat{X} , \hat{Y} , and \hat{Z} are bath operators. The off-diagonal coupling ($\hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y}$) induces population relaxation. The diagonal coupling $\hat{\sigma}_z \otimes \hat{Z}$ induces pure dephasing.

2.1 Carr–Purcell–Meiboom–Gill DD

For the sake of simplicity, we first consider the pure dephasing case ($\hat{X} = \hat{Y} = 0$). In the absence of controlling pulses, the evolution of the quantum state $|\Psi(\tau)\rangle = \hat{U}_0 |\Psi(0)\rangle$ of the coupled qubit-bath system is driven by the free propagator $\hat{U}_0 \equiv e^{-i\hat{H}\tau} = e^{-i(\hat{C} + \hat{\sigma}_z \otimes \hat{Z})\tau}$.

The Hahn echo [20] is realized by a single instantaneous π pulse applied at the middle of the evolution to switch the qubit states between $|\uparrow\rangle$ and $|\downarrow\rangle$,

$$|\Psi(2\tau)\rangle = \hat{U}_0 \hat{\sigma}_x \hat{U}_0 |\Psi(0)\rangle \quad (12)$$

so that the propagator for the whole evolution from 0 to 2τ is $\hat{U}_0 \hat{\sigma}_x \hat{U}_0 = \hat{\sigma}_x \hat{U}_1$ with

$$\begin{aligned} \hat{U}_1 &\equiv \hat{\sigma}_x \hat{U}_0 \hat{\sigma}_x \hat{U}_0 = e^{-i(\hat{C} - \hat{\sigma}_z \otimes \hat{Z})\tau} e^{-i(\hat{C} + \hat{\sigma}_z \otimes \hat{Z})\tau} \\ &= e^{-i2\tau[\hat{C} + \hat{\sigma}_z \otimes \hat{Z} \cdot O(\hat{C}\tau)]} \end{aligned} \quad (13)$$

In the propagator, the qubit–bath coupling is eliminated in the first order of the pulse interval τ . By repeating the Hahn echo propagator \hat{U}_1 , the Carr–Purcell–Meiboom–Gill DD (CPMG) [45, 46] can be constructed so as to preserve the coherence of the qubit for a long time.

The building block of CPMG consists of two instantaneous π pulses applied at τ and 3τ , respectively. At the end of the evolution $t = 4\tau$, the state of the qubit-bath system is $|\Psi(4\tau)\rangle = \hat{U}_2 |\Psi(0)\rangle$, where the propagator

$$\begin{aligned} \hat{U}_2 &= \hat{U}_0 \hat{\sigma}_x \hat{U}_0 \hat{U}_0 \hat{\sigma}_x \hat{U}_0 = \hat{\sigma}_x \hat{U}_1 \hat{\sigma}_x \hat{U}_1 \\ &= e^{-i4\tau[\hat{C} + \hat{\sigma}_z \otimes \hat{Z} \cdot O(\hat{C}^2\tau^2)]} \end{aligned} \quad (14)$$

is obtained by embedding \hat{U}_1 into the basic structure $\hat{\sigma}_x(\dots)\hat{\sigma}_x(\dots)$. The CPMG sequence of $2N$ pulses is obtained by repeating the building block \hat{U}_2 for N times. The propagator for the whole evolution from 0 to $T = 4N\tau$ is

$$\hat{U}_{\text{CPMG}} = \hat{U}_2^N = e^{-iT[\hat{C} + \hat{\sigma}_z \otimes \hat{Z} \cdot O(\hat{C}^2\tau^2)]} \quad (15)$$

The qubit–bath coupling is eliminated up to the second order of the minimum pulse interval τ .

2.2 Concatenated DD

Note that in Eq. (14), the building unit of CPMG can

be viewed as a nested application of the Hahn echo, which eliminates the qubit–bath coupling to one order higher than the simple Hahn echo does. It was noticed in Refs. [26, 27] that a mirror-symmetric arrangement of two DD sequences can decouple a quantum object to a higher order. Furthermore, Ref. [27] mentioned the possibility of realizing DD to an arbitrary order by iterative construction. Khodjasteh and Lidar proposed the first explicit concatenated DD (CDD) scheme [30, 31] to eliminate arbitrary qubit–bath coupling (including both diagonal and off-diagonal couplings) with an intuitive geometrical understanding [47]. The idea of CDD was further developed by the incorporation of randomness into the sequence for improvement of performance [32, 36]. CDD schemes against pure dephasing were investigated for electron spin qubits in realistic solid-state systems with nuclear spins as baths [33, 35]. The advantage of CDD over the periodic DD sequences has been observed in experiments for nuclear spin qubits in solid-state environments [43].

The propagator for CDD is obtained by recursion

$$\hat{U}_n = \hat{\sigma}_x \hat{U}_{n-1} \hat{\sigma}_x \hat{U}_{n-1} = e^{-i2^n \tau [\hat{C} + \hat{\sigma}_z \otimes \hat{Z} \cdot O(\hat{C}^n \tau^n)]} \quad (16)$$

in which the qubit–bath coupling has been eliminated up to the n th order of the minimum pulse interval τ . By increasing the concatenation level n , the qubit–bath coupling can be eliminated up to an arbitrary order of τ .

For the most general qubit-bath Hamiltonian in Eq. (11), the idea of concatenation can still be applied to eliminate both the pure dephasing term $\hat{\sigma}_z \otimes \hat{Z}$ and the relaxation term $\hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y}$. In the absence of controlling pulses, the evolution of the qubit-bath system is driven by the free propagator $\hat{U}_0 \equiv e^{-i\hat{H}\tau}$. The qubit-bath coupling can be eliminated up to the first order of τ by the controlled evolution [30]:

$$\begin{aligned} \hat{U}_1 &\equiv \hat{U}_0 \left(\hat{\sigma}_x \hat{U}_0 \hat{\sigma}_x \right) \left(\hat{\sigma}_y \hat{U}_0 \hat{\sigma}_y \right) \left(\hat{\sigma}_z \hat{U}_0 \hat{\sigma}_z \right) \\ &= e^{-i\tau(\hat{C} + \hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y} + \hat{\sigma}_z \otimes \hat{Z})} \\ &\quad \cdot e^{-i\tau(\hat{C} + \hat{\sigma}_x \otimes \hat{X} - \hat{\sigma}_y \otimes \hat{Y} - \hat{\sigma}_z \otimes \hat{Z})} \\ &\quad \cdot e^{-i\tau(\hat{C} - \hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y} - \hat{\sigma}_z \otimes \hat{Z})} \\ &\quad \cdot e^{-i\tau(\hat{C} - \hat{\sigma}_x \otimes \hat{X} - \hat{\sigma}_y \otimes \hat{Y} + \hat{\sigma}_z \otimes \hat{Z})} \\ &= e^{-i4\tau[\hat{C} + O(\alpha\beta\tau)]} \end{aligned} \quad (17)$$

where α denotes the norm of \hat{C} , and β denotes the norm of $\hat{X}, \hat{Y}, \hat{Z}$. Thus, all the qubit–bath coupling terms are eliminated in the first order of the minimum pulse interval τ . By concatenation, the propagator for the n th-order CDD is

$$\begin{aligned} \hat{U}_n &\equiv \hat{U}_{n-1} \left(\hat{\sigma}_x \hat{U}_{n-1} \hat{\sigma}_x \right) \left(\hat{\sigma}_y \hat{U}_{n-1} \hat{\sigma}_y \right) \left(\hat{\sigma}_z \hat{U}_{n-1} \hat{\sigma}_z \right) \\ &= e^{-i4^n \tau [\hat{C} + O(\alpha\beta^n \tau^n) + O(\alpha^2 \beta^{n-1} \tau^n) + \dots + O(\alpha^n \beta \tau^n)]} \end{aligned} \quad (18)$$

in which the qubit-bath coupling has been eliminated up to the n th order of τ . By increasing the concatenation

level n , the qubit–bath coupling can be eliminated up to an arbitrary order.

To eliminate the qubit-bath coupling to a given order N of the evolution time, the number of instantaneous π pulses scales exponentially with the order of CDD, namely, $N_{\text{pulse}} = O(2^N)$ for eliminating the pure dephasing term and $N_{\text{pulse}} = O(4^N)$ for eliminating all qubit-bath couplings. Since errors are inevitably introduced in each π pulse, it is desirable to minimize the number of controlling pulses used to achieve a given order of decoupling.

2.3 Uhrig DD

Uhrig DD (UDD) [37, 48–50] is a remarkable advance in the DD theory. UDD can eliminate the qubit-bath pure dephasing up to the N th order of the evolution time using N instantaneous π pulses applied at

$$T_j = T \sin^2 \frac{j\pi}{2(N+1)}, \quad j = 1, 2, \dots, N \quad (19)$$

during the evolution of the qubit-bath system from 0 to T . UDD is optimal in the sense that it uses the minimum number of control pulses for a given order of decoupling. Such pulse sequences for $N \leq 5$ were first noticed by Dhar *et al.* in designing control of the quantum Zeno effect [51]. The application of such sequences to DD was first proposed by Uhrig for a pure dephasing spin-boson model [37]. Then, Lee, Witzel and Das Sarma conjectured that UDD may work for a general pure dephasing model with an analytical verification up to $N = 9$ [48]. Later, computer-assisted algebra was used to verify the conjecture up to $N = 14$ [49]. Finally, UDD was rigorously proved to be universal for any order N and was also extended to the case of population relaxation [50]. The performance bounds for UDD against pure dephasing were also established [52].

By construction, UDD is optimal for a finite system (or a system with a hard cut-off in the spectrum) in the “high fidelity” regime where a short-time expansion of the qubit–bath propagator converges. For the “low fidelity” regime, further theoretical work [38, 53] shows that UDD is optimal when the noise spectrum has a hard cutoff, while CPMG performs better than CDD and UDD when the noise has a soft cutoff or when the hard cutoff is not reached by the spectrum filtering functions corresponding to the DD sequences. The experimental investigations of UDD were carried out in an array of ~ 1000 Be⁺ ions [40] and in irradiated malonic acid crystals [41].

2.3.1 Spin-boson model: Discovery of UDD

The qubit–bath Hamiltonian \hat{H}_{sb} of the spin-boson pure dephasing model [37] corresponds to $\hat{C} = \sum_i \omega_i \hat{b}_i^\dagger \hat{b}_i$ and

$\hat{Z} = \sum_i (\kappa_i/2)(\hat{b}_i^\dagger + \hat{b}_i)$ in Eq. (7), where \hat{b}_i is the bosonic annihilation operator. For N instantaneous π pulses applied at $T_1, T_2, \dots, T_N \in [0, T]$, the propagator for the evolution from 0 to T is

$$\hat{U}(T, 0) = \hat{U}_0(T - T_N) \hat{\sigma}_x \hat{U}_0(T_N - T_{N-1}) \cdots \cdot \hat{\sigma}_x \hat{U}_0(T_2 - T_1) \hat{\sigma}_x \hat{U}_0(T_1) \quad (20)$$

where $\hat{U}_0(t) = e^{-i\hat{H}_{sb}t}$ is the free propagator. $\hat{U}(T, 0)$ can be written as \hat{U}_N (for N being even) or $\hat{\sigma}_x \hat{U}_N$ (for N being odd) with

$$\begin{aligned} \hat{U}_N &= e^{-i(\hat{C} + (-1)^N \hat{\sigma}_z \otimes \hat{Z})(T - T_N)} \\ &\cdot e^{-i(\hat{C} + (-1)^{N-1} \hat{\sigma}_z \otimes \hat{Z})(T_N - T_{N-1})} \cdots \\ &\cdot e^{-i(\hat{C} - \hat{\sigma}_z \otimes \hat{Z})(T_2 - T_1)} e^{-i(\hat{C} + \hat{\sigma}_z \otimes \hat{Z})T_1} \\ &= \frac{\hat{U}_N^{(+)} + \hat{U}_N^{(-)}}{2} + \hat{\sigma}_z \otimes \frac{\hat{U}_N^{(+)} - \hat{U}_N^{(-)}}{2} \end{aligned} \quad (21)$$

where $\hat{U}_N^{(\pm)}$ is obtained from \hat{U}_N by replacing $\hat{\sigma}_z$ by ± 1 . In the N th-order UDD, the positions T_1, T_2, \dots, T_N of the N pulses are fixed by requiring that in the propagator \hat{U}_N , the qubit-bath coupling should be eliminated up to the N th order, i.e.,

$$\begin{aligned} \delta\hat{U}(T) &\equiv \hat{U}_N^{(+)} - \hat{U}_N^{(-)} = \hat{U}_N^{(-)} [(\hat{U}_N^{(-)})^\dagger \hat{U}_N^{(+)} - 1] \\ &\sim O(T^{N+1}) \end{aligned} \quad (22)$$

or equivalently,

$$\tilde{\delta}\hat{U}(T) \equiv (\hat{U}_N^{(-)})^\dagger \hat{U}_N^{(+)} - 1 \sim O(T^{N+1}) \quad (23)$$

By exact diagonalization of the spin-boson Hamiltonian, $\tilde{\delta}\hat{U}(T)$ has been evaluated as $\tilde{\delta}\hat{U}(T) = e^{2\hat{\Delta}(T)}$ with

$$\hat{\Delta}(T) = \sum_{j=0}^N (-1)^j \left[\hat{K}_I(T_j) - \hat{K}_I(T_{j+1}) \right] \quad (24)$$

where we have defined $T_0 \equiv 0$, $T_{N+1} \equiv T$, and

$$\hat{K}_I(t) \equiv e^{i\hat{C}t} \hat{K} e^{-i\hat{C}t} \quad (25a)$$

$$\hat{K} \equiv \sum_i \frac{\kappa_i}{2\omega_i} (\hat{b}_i^\dagger - \hat{b}_i) \quad (25b)$$

The Taylor expansion

$$\begin{aligned} \hat{K}_I(t) &= \hat{K} + \sum_{p=1}^{\infty} \frac{(it)^p}{p!} \overbrace{\left[\hat{C}, \dots, \left[\hat{C}, \left[\hat{C}, \hat{K} \right] \right] \right]}^{p\text{-fold commutator}} \\ &\equiv \hat{K} + \sum_{p=1}^{\infty} \hat{K}_p t^p \end{aligned} \quad (26)$$

yields $\hat{\Delta}(T) = - \sum_{p=1}^{\infty} \hat{K}_p T^p A_p$, where

$$A_p \equiv \sum_{j=0}^N (-1)^j \left[\left(\frac{T_{j+1}}{T} \right)^p - \left(\frac{T_j}{T} \right)^p \right] \quad (27)$$

Thus, the condition $\tilde{\delta}\hat{U}(T) = O(T^{N+1})$ is equivalent to

N coupled algebra equations

$$A_p = 0, \quad p = 1, 2, \dots, N \quad (28)$$

whose unique physical solution is the UDD sequence in Eq. (19). The UDD sequence is optimal in that it uses the minimum number of pulses to make the first N terms of A_p 's vanish and eliminate the qubit-bath coupling up to the N th order.

2.3.2 Geometrical interpretation of decoherence and DD

Here, we give a geometrical interpretation of decoherence and DD by considering the spin-boson pure dephasing model, based on trajectories of bath quantum states in the Hilbert space conditioned on the qubit states and DD control. The pure dephasing qubit-bath Hamiltonian can be reformulated as:

$$\hat{H} \equiv \sum_{\pm} |\pm\rangle \langle \pm| \otimes \hat{H}_{\pm} \quad (29)$$

where $\{|\pm\rangle\}$ denote the two eigenstates of the qubit, and the bath operators $\hat{H}_{\pm} = \hat{C} \pm \hat{Z}$. The qubit coherence is given by the overlap of bath states, as shown in Eq. (10).

The state of the bosonic bath can be described in the basis of coherent states [54]. The coherent state of the l th boson mode is $|P_l\rangle \equiv e^{P_l \hat{b}_l^\dagger - P_l^* \hat{b}_l} |0\rangle$ with P_l being a complex number. A coherent state $|P_l(t_0)\rangle$ after a time of evolution under the Hamiltonians $H_{l,\pm} = \omega_l \hat{b}_l^\dagger \hat{b}_l \pm \frac{1}{2} \kappa_l (\hat{b}_l^\dagger + \hat{b}_l)$ is still a coherent state:

$$e^{-i\hat{H}_{l,\pm}(t-T_0)} |P_l(T_0)\rangle = |P_{l,\pm}(t)\rangle e^{-i\theta_{\pm}} \quad (30)$$

where

$$P_{l,\pm}(t) = \left[P_l(T_0) - (\mp \frac{\kappa_l}{2\omega_l}) \right] e^{-i\omega_l(t-T_0)} \mp \frac{\kappa_l}{2\omega_l} \quad (31)$$

and the phase factor $\theta_{\pm}(t) = \pm \int_{T_0}^t \Re[\frac{1}{2} \kappa_l P_l^*(t)] dt$.

As illustrated in Fig. 1, the complex numbers $P_{l,\pm}(t)$, which represent the coherent states, are rotating clockwise about the points $\mp \frac{\kappa_l}{2\omega_l}$ in the complex plane with an amplitude of $|P_l(T_0)|$ and an angular frequency ω_l . The overlap of the bifurcated states

$$|\langle P_{l,+}(t) | P_{l,-}(t) \rangle| = \exp(-|P_{l,+}(t) - P_{l,-}(t)|^2) \quad (32)$$

is determined by their distance in the complex plane. We consider the case that the initial bath state is a coherent state $|J\rangle = \otimes_l |P_l(T_0)\rangle$. Thus, at time t , the qubit coherence is

$$L_J(t) = \bigotimes_l |\langle P_{l,+}(t) | P_{l,-}(t) \rangle| \quad (33)$$

which decreases when the distance between $P_{l,+}(t)$ and $P_{l,-}(t)$ in the complex plane is increased. Since the bifurcated evolution of the bath is determined by the

qubit states $|\pm\rangle$, during the qubit-bath evolution, instantaneous flips of the qubit states will cause the bath evolution pathways to exchange their rotation centers. At some later time, the two bifurcated pathways could cross into each other, upon which the qubit and the boson mode become disentangled. At this disentanglement point, the which-way information is erased, and therefore, the qubit coherence is recovered.

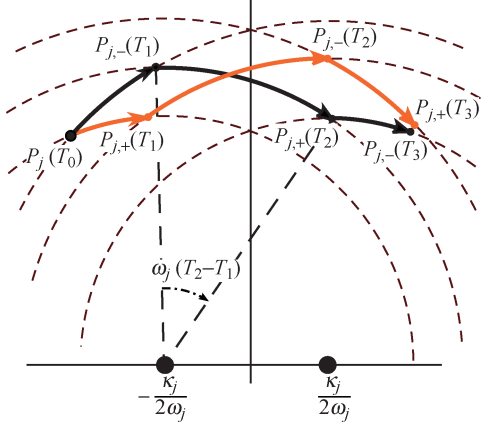


Fig. 1 The bifurcated trajectories (solid curves) of $P_{j,\pm}(t)$ in the complex plane under qubit flip control applied at $T_1 = T/4$ and $T_2 = 3T/4$, with the initial time $T_0 = 0$ and final time $T_3 = T$. The arrows on the solid curves indicate the evolution directions.

Let the bifurcated bath states at time T_{m-1} be denoted by the complex numbers $\{P_{l,\mp}(T_{m-1})\}$. Suppose there is a qubit flip applied at $t = T_{m-1}$. After an interval of evolution, the bath states will become

$$P_{l,\pm}(T_m) = [P_{l,\mp}(T_{m-1}) \pm \frac{\kappa_l}{2\omega_l}] e^{-i\omega_l(T_m - T_{m-1})} \mp \frac{\kappa_l}{2\omega_l} \quad (34)$$

We define the difference $\Delta_m^l \equiv P_{l,+}(T_m) - P_{l,-}(T_m)$. By recursively using the initial condition

$$P_{l,+}(T_0) = P_{l,-}(T_0) = P_l(T_0) \quad (35)$$

and Eq. (34), we have that after N flips at times T_1, T_2, \dots, T_N , the difference

$$\Delta_{N+1}^l = i(-1)^{N+1} e^{-i\omega_l T_{N+1}} \kappa_l f(\omega_l) \quad (36)$$

with

$$f(\omega_l) \equiv \frac{1}{i\omega_l} \sum_{j=0}^N (-1)^j (e^{i\omega_l T_{j+1}} - e^{i\omega_l T_j}) \quad (37)$$

Eqs. (34) and (36) give us a geometrical interpretation of control of decoherence by qubit flips. In Fig. 1, we show the evolution of $P_{j,\pm}(T_m)$ for qubit flips occurring at $T_1 = T/4$ and $T_2 = 3T/4$, with the total evolution time $T_3 = T$.

Note that the initial $P_l(T_0)$ is canceled in the expression of Δ_{N+1}^l in Eq. (36). Thus, from Eqs. (32) and (33), the coherence L_J is independent of the initial bath state

$|J\rangle = \otimes_l |P_l(T_0)\rangle$. By the expansion of $f(\omega_l)$, we obtain

$$\Delta_{N+1}^l = (-1)^{N+1} e^{-i\omega_l T} \frac{\kappa_l}{\omega_l} \sum_{n=1}^{\infty} \frac{(i\omega_l T)^n}{n!} \Lambda_n \quad (38)$$

where Λ_n is given by Eq. (27). The distance Δ_{N+1}^l between $P_{l,\pm}(T_{N+1})$ is a small quantity $\sim O(T^{N+1})$ if $\{\Lambda_n = 0\}$ for $n \leq N$. Thus, the conditions for UDD are reproduced.

2.3.3 Proof of universality of UDD against pure dephasing

The proof of the universality (i.e., model independence) [50] of UDD is facilitated by the observation that to eliminate the qubit-bath coupling to a given order, one needs only to eliminate the odd-power terms of the coupling $\hat{\sigma}_z \otimes \hat{Z}$ in the perturbative expansion of the propagator, since the even-power terms of $\hat{\sigma}_z \otimes \hat{Z}$ is a pure bath operator, $(\hat{\sigma}_z \otimes \hat{Z})^{2m} = \hat{Z}^{2m}$, which does not cause qubit decoherence. We will present the proof in the interaction picture following Ref. [50], which can be easily reformulated in other pictures [52].

As discussed in the previous section, for the pure dephasing Hamiltonian in Eq. (7) under the control of the N th-order UDD sequence, the propagator from 0 to T is given by

$$\begin{aligned} \hat{U}_N &= e^{-i(\hat{C} + (-1)^N \hat{\sigma}_z \otimes \hat{Z})(T - T_N)} \\ &\cdot e^{-i(\hat{C} + (-1)^{N-1} \hat{\sigma}_z \otimes \hat{Z})(T_N - T_{N-1})} \dots \\ &\cdot e^{-i(\hat{C} - \hat{\sigma}_z \otimes \hat{Z})(T_2 - T_1)} e^{-i(\hat{C} + \hat{\sigma}_z \otimes \hat{Z})T_1} \end{aligned} \quad (39)$$

the proof of the universality of UDD is equivalent to proving

$$\hat{U}_N = \hat{U}_N^{(\text{bath})} + O(T^{N+1}) \quad (40)$$

where $\hat{U}_N^{(\text{bath})}$ is a bath operator containing no qubit operators. With the standard perturbation theory in the interaction picture, Eq. (39) can be put in the time-ordered formal expression:

$$\hat{U}_N = e^{-i\hat{C}T} \hat{T} e^{-i \int_0^T F_N(t) \hat{\sigma}_z \otimes \hat{Z}_I(t) dt} \quad (41)$$

where \hat{T} is the time-ordering operator, the modulation function $F_N(t) \equiv (-1)^j$ for $t \in [T_j, T_{j+1}]$ with $T_0 \equiv 0$ and $T_{N+1} \equiv T$, and

$$\begin{aligned} \hat{Z}_I(t) &\equiv e^{i\hat{C}t} \hat{Z} e^{-i\hat{C}t} = \sum_{p=0}^{\infty} \frac{(it)^p}{p!} \overbrace{[\hat{C}, [\hat{C}, \dots [\hat{C}, \hat{Z}] \dots]]}^{p\text{-folds commutator}} \\ &\equiv \sum_{p=0}^{\infty} \hat{Z}_p t^p \end{aligned} \quad (42)$$

The propagator can be expanded into Taylor series:

$$\hat{U}_N = e^{-i\hat{C}T} \sum_{n=0}^{\infty} (-i\hat{\sigma}_z)^n \otimes \hat{\Delta}_n \equiv \hat{U}_N^{(\text{even})} + \hat{U}_N^{(\text{odd})} \quad (43)$$

where

$$\hat{\Delta}_n \equiv \int_0^T F_N(t_n) dt_n \int_0^{t_n} F_N(t_{n-1}) dt_{n-1} \cdots \cdot \int_0^{t_2} F_N(t_1) dt_1 \hat{Z}_I(t_n) \hat{Z}_I(t_{n-1}) \cdots \hat{Z}_I(t_1) \quad (44)$$

is a pure bath operator. Here,

$$\hat{U}_N^{(\text{even})} = e^{-i\hat{C}T} \sum_{k=0}^{\infty} (-i)^{2k} \hat{\Delta}_{2k} \quad (45)$$

consists of even powers of the qubit-bath coupling $\hat{\sigma}_z \otimes \hat{Z}$ and therefore is a pure bath operator, which does not induce qubit dephasing. The term consisting of the odd powers of the qubit bath coupling

$$\hat{U}_N^{(\text{odd})} = \hat{\sigma}_z \otimes e^{-i\hat{C}T} \sum_{k=0}^{\infty} (-i)^{2k+1} \hat{\Delta}_{2k+1} \quad (46)$$

induces the qubit dephasing. We just need to show that $\hat{\Delta}_{2k+1} = O(T^{N+1})$.

Using the expansion in Eq. (42), we have

$$\hat{\Delta}_n = \sum_{\{p_j\}} \left(\hat{Z}_{p_n} \cdots \hat{Z}_{p_2} \hat{Z}_{p_1} F_{p_1, p_2, \dots, p_n} T^{n+p_1+p_2+\dots+p_n} \right) \quad (47)$$

where

$$F_{p_1, \dots, p_n} \equiv \int_0^T \frac{dt_n}{T} \cdots \int_0^{t_3} \frac{dt_2}{T} \int_0^{t_2} \frac{dt_1}{T} \cdot \prod_{j=1}^n F_N(t_j) \left(\frac{t_j}{T} \right)^{p_j} \quad (48)$$

is a dimensionless constant independent of T . Now, the problem is reduced to proving

$$F_{p_1, p_2, \dots, p_n} = 0 \quad (49)$$

for n being odd and $n + \sum_{j=1}^n p_j \leq N$. For this purpose, we make the variable substitution $t_j = T \sin^2(\theta_j/2)$ and define the scaled modulation function:

$$f_N(\theta) \equiv F_N \left(T \sin^2 \frac{\theta}{2} \right) = (-1)^j \quad (50)$$

for $\theta \in [j\pi/(N+1), (j+1)\pi/(N+1)]$. With

$$\sin^{2p} \frac{\theta}{2} \sin \theta = (2i)^{-2p} \sum_{r=0}^{2p} C_{2p}^r \sin[(p-r+1)\theta] \quad (51)$$

we can write F_{p_1, p_2, \dots, p_n} as a linear combination of terms in the form

$$f_{q_1, \dots, q_n} \equiv \int_0^\pi d\theta_n \cdots \int_0^{\theta_3} d\theta_2 \int_0^{\theta_2} d\theta_1 \cdot \prod_{j=1}^n f_N(\theta_j) \sin(q_j \theta_j) \quad (52)$$

with $|q_j| \leq p_j + 1$. It suffices to show that $f_{q_1, q_2, \dots, q_n} = 0$ for odd n and $\sum_{j=1}^n |q_j| \leq N$. We notice that $f_N(\theta)$ is a periodic function with a period of $2\pi/(N+1)$, and

therefore, can be expanded into Fourier series:

$$f_N(\theta) = \sum_{k=1,3,5,\dots} \frac{4}{k\pi} \sin[k(N+1)\theta] \quad (53)$$

The key feature of the Fourier expansion to be exploited is that it contains only odd harmonics of $\sin[(N+1)\theta]$. With the Fourier expansion, we just need to show that

$$\int_0^\pi d\theta_n \cdots \int_0^{\theta_3} d\theta_2 \int_0^{\theta_2} d\theta_1 \prod_{j=1}^n \cos(r_j \theta_j + q_j \theta_j) = 0 \quad (54)$$

for n being odd, r_j being an odd multiple of $(N+1)$, and $\sum_{j=1}^n |q_j| \leq N$. With the product-to-sum trigonometric formula repeatedly used, it can be shown by induction that after an even number of variables $\theta_1, \theta_2, \dots, \theta_{2k}$ have been integrated over, the resultant integrand as a function of θ_{2k+1} is the sum of cosine functions of the form

$$\cos(R_{2k+1}\theta_{2k+1} + Q_{2k+1}\theta_{2k+1}) \quad (55)$$

with R_{2k+1} being an odd multiple of $(N+1)$ and $|Q_{2k+1}| \leq \sum_{j=1}^{2k+1} |q_j|$. In particular, the last step is

$$\int_0^\pi \cos(R_n \theta_n + Q_n \theta_n) d\theta_n \quad (56)$$

Since R_n is an odd (non-zero, of course) multiple of $(N+1)$, and $|Q_n| \leq \sum_{j=1}^n |q_j| \leq N$, we have $R_n + Q_n \neq 0$, and the integral above must be zero. Thus, Eq. (49) holds. The proof is done.

It should be noted that in the proof above, the perturbation-theoretical expansion requires that the Hamiltonian of the bath has a finite norm, which means that the noise spectrum felt by the qubit has a hard cut-off.

2.3.4 Universality of UDD against population relaxation

A straightforward corollary of Eq. (49) is that UDD can also be used to suppress population relaxation of the qubit. Considering the most general qubit-bath Hamiltonian in Eq. (11) and assuming that the UDD sequence consists of N instantaneous π pulses to rotate the qubit around the z -axis, we aim to show that the relaxation of the qubit population in $|\uparrow\rangle$ and $|\downarrow\rangle$ is eliminated up to $O(T^N)$. The propagator of the qubit-bath evolution from 0 to T is

$$\hat{U}(T, 0) = \hat{U}_0(T - T_N) \hat{\sigma}_z \hat{U}_0(T_N - T_{N-1}) \cdots \cdot \hat{\sigma}_z \hat{U}_0(T_2 - T_1) \hat{\sigma}_z \hat{U}_0(T_1) \quad (57)$$

where $\hat{U}_0(t) = e^{-i\hat{H}t}$ is the free propagator. $\hat{U}(T, 0)$ can be written as \hat{U}_N (for N being odd) or $\hat{\sigma}_z \hat{U}_N$ (for N being even) with

$$\hat{U}_N = e^{-i[\hat{C}' + (-1)^N \hat{D}](T - T_N)} \dots$$

$$\cdot e^{-i(\hat{C}' - \hat{D})(T_2 - T_1)} e^{-i(\hat{C}' + \hat{D})T_1} \quad (58)$$

in which the Hamiltonian has been separated into $\hat{C}' \equiv \hat{C} + \hat{\sigma}_z \otimes \hat{Z}$ and $\hat{D} \equiv \hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y}$. With the definition $\hat{D}_I(t) \equiv e^{i\hat{C}'t} \hat{D} e^{-i\hat{C}'t}$, the propagator can be formally expressed as:

$$\hat{U}_N = e^{-i\hat{C}'T} \hat{T} e^{-i \int_0^T F_N(t) \hat{D}_I(t) dt} = \hat{U}_N^{(\text{even})} + \hat{U}_N^{(\text{odd})} \quad (59)$$

where

$$\hat{U}_N^{(\text{even})} = e^{-i\hat{C}'T} \sum_{k=0}^{\infty} (-i)^{2k} \hat{\Delta}'_{2k} \quad (60)$$

consists of even powers of \hat{D} , and

$$\hat{U}_N^{(\text{odd})} = e^{-i\hat{C}'T} \sum_{k=0}^{\infty} (-i)^{2k+1} \hat{\Delta}'_{2k+1} \quad (61)$$

consists of odd powers of \hat{D} , with $\hat{\Delta}'_n$ obtained from $\hat{\Delta}_n$ in Eq. (44) by replacing $\hat{Z}_I(t)$ by $\hat{D}_I(t)$. By expanding $\hat{D}_I(t)$ into Taylor series [similar to Eq. (42)]

$$\hat{D}_I(t) = \sum_{p=0}^{\infty} \hat{D}_p t^p \quad (62)$$

the identity Eq. (49) immediately gives $\hat{\Delta}'_{2k+1} = O(T^{N+1})$. As a result, $\hat{U}_N^{(\text{odd})} = O(T^{N+1})$ and the propagator

$$\hat{U}_N = \hat{U}_N^{(\text{even})} + O(T^{N+1}) \quad (63)$$

contains only even powers of \hat{D} up to $O(T^N)$. Since \hat{D} contains only the Pauli matrices $\hat{\sigma}_x$ and $\hat{\sigma}_y$ and an even power of the two Pauli matrices $\hat{\sigma}_x^{n_x} \hat{\sigma}_y^{n_y}$ (with $n_x + n_y$ being even) is either unity or $i\hat{\sigma}_z$, the propagator

$$\hat{U}_N = e^{-i\hat{H}_{\text{eff}}(T)T + O(T^{N+1})} \quad (64)$$

where the effective Hamiltonian $\hat{H}_{\text{eff}}(T) = \hat{C}_{\text{eff}}(T) + \hat{\sigma}_z \otimes \hat{Z}_{\text{eff}}(T)$ contains only pure dephasing term $\hat{\sigma}_z \otimes \hat{Z}_{\text{eff}}(T)$ and commutes with $\hat{\sigma}_z$. Thus, the N -pulse UDD eliminates the population relaxation up to $O(T^N)$.

2.3.5 Time-dependent Hamiltonians

From the procedures following Eqs. (42) and (62), it is immediately observed that the proof above applies to time-dependent Hamiltonian as long as a Taylor expansion of the Hamiltonian similar to those in Eqs. (42) and (62) exists (such as a Hamiltonian having analytical time dependence). Such a generalization was presented by Pasini and Uhrig [55].

2.3.6 UDD with non-instantaneous pulses

With the help of Eq. (54), we realize that Eq. (49) holds for more general modulation functions $F_N(t)$ as long as the scaled modulation function

$f_N(\theta) \equiv F_N(T \sin^2(\theta/2))$ contains only odd harmonics of $\sin[(N+1)\theta]$ as in Eq. (53), i.e.,

$$f_N(\theta) = \sum_{k=0}^{\infty} A_k \sin[(2k+1)(N+1)\theta] \quad (65)$$

with arbitrary coefficients A_k . Motivated by this observation, we try to generalize UDD to the case of π pulses with a finite duration.

For the case of UDD against general decoherence, we consider the control of the qubit by an arbitrary time-dependent magnetic field $B(t)$ applied along a certain direction to protect the qubit coherence along this axis. Without loss of generality, we take this direction as the z -axis. The general qubit-bath Hamiltonian under DD control is

$$\hat{H}(t) = \hat{C} + \hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y} + \hat{\sigma}_z \otimes \hat{Z} + \frac{1}{2} \hat{\sigma}_z B(t) \quad (66)$$

In the rotating reference frame following the qubit precession under the magnetic field, the Hamiltonian becomes

$$\hat{H}_R(t) = \hat{C}' + \cos[\phi(t)] \hat{D}^+ + \sin[\phi(t)] \hat{D}^- \quad (67)$$

where the precession angle $\phi(t) = \int_0^t B(t') dt'$, $\hat{C}' \equiv \hat{C} + \hat{\sigma}_z \otimes \hat{Z}$, $\hat{D}^+ \equiv \hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y}$, and $\hat{D}^- \equiv \hat{\sigma}_x \otimes \hat{Y} - \hat{\sigma}_y \otimes \hat{X}$. The propagator in the rotating reference frame is

$$\hat{U}_N = e^{-i\hat{C}'T} \hat{T} \exp \left(-i \int_0^T \sum_{\lambda=\pm} F_N^\lambda(t) D_I^\lambda(t) dt \right) \quad (68)$$

with $F_N^+(t) = \cos[\phi(t)]$, $F_N^-(t) = \sin[\phi(t)]$, and $\hat{D}_I^\lambda(t) = e^{i\hat{C}'t} \hat{D}^\lambda e^{-i\hat{C}'t}$. Again, we decompose \hat{U}_N as the sum of $\hat{U}_N^{(\text{even})}$ (which consists of even powers of \hat{D}^\pm) and $\hat{U}_N^{(\text{odd})}$ (which consists of odd powers of \hat{D}^\pm),

$$\hat{U}_N^{(\text{even})} = e^{-i\hat{C}'T} \sum_{k=0}^{\infty} \sum_{\lambda_1, \dots, \lambda_{2k}} (-i)^{2k} \hat{\Delta}_{2k}^{(\lambda_1, \dots, \lambda_{2k})} \quad (69a)$$

$$\hat{U}_N^{(\text{odd})} = e^{-i\hat{C}'T} \sum_{k=0}^{\infty} \sum_{\lambda_1, \dots, \lambda_{2k+1}} (-i)^{2k+1} \hat{\Delta}_{2k+1}^{(\lambda_1, \dots, \lambda_{2k+1})} \quad (69b)$$

where

$$\begin{aligned} \hat{\Delta}_n^{(\lambda_1, \dots, \lambda_n)} &\equiv \int_0^T F_N^{\lambda_n}(t_n) dt_n \int_0^{t_n} F_N^{\lambda_{n-1}}(t_{n-1}) dt_{n-1} \\ &\cdot \dots \int_0^{t_2} F_N^{\lambda_1}(t_1) dt_1 \hat{D}_I^{\lambda_n}(t_n) \hat{D}_I^{\lambda_{n-1}}(t_{n-1}) \\ &\cdot \dots \hat{D}_I^{\lambda_1}(t_1) \end{aligned} \quad (70)$$

has a structure similar to $\hat{\Delta}_n$ in Eq. (44). After expanding $\{\hat{D}_I^{\lambda_k}(t_k)\}$ into Taylor series, the identity Eq. (49) immediately gives $\hat{\Delta}_{2k+1}^{(\lambda_1, \dots, \lambda_{2k+1})} = O(T^{N+1})$. Thus, the qubit decoherence along the z -axis is suppressed to $O(T^{N+1})$ as long as the scaled modulation function $f_N^\pm(\theta) \equiv F_N^\pm(T \sin^2(\theta/2))$ contains only odd harmonics of $\sin[(N+1)\theta]$ as depicted in Eq. (65). This condition is satisfied if and only if the scaled modulation functions

$f_N^\pm(\theta)$ have the following symmetries:

- 1) periodic with period of $2\pi/(N+1)$;
- 2) anti-symmetric with respect to $\theta = j\pi/(N+1)$;
- 3) symmetric with respect to $\theta = (j+1/2)\pi/(N+1)$.

The anti-symmetry condition requires $f_N^\pm(\theta)$ be either zero or discontinuous at $\theta = j\pi/(N+1)$. However, $f_N^+(\theta)$ and $f_N^-(\theta)$ cannot be simultaneously zero since they have to satisfy the normalization condition

$$[f_N^+(\theta)]^2 + [f_N^-(\theta)]^2 = 1 \quad (71)$$

according to the definition of $F_N^\pm(t)$. So there must be sudden jumps at least in one of two modulation functions at $\theta = j\pi/(N+1)$, which means that the controlling magnetic field $B(t)$ has to contain a δ -pulse for π -rotation at $t = T_j$. With the initial conditions $f_N^+(0) = 1$ and $f_N^-(0) = 0$, one can choose the field such that $f_N^-(\theta)$ is continuous, while $f_N^+(\theta)$ has sudden jumps between +1 and -1 at $\theta = j\pi/(N+1)$. Thus, a generalized UDD sequence can be chosen the following way: For $\theta \in [0, \pi/(2N+2)]$, $f_N^+(\theta)$ can be arbitrary but sudden jumps from -1 to +1 at $\theta = 0$ and from +1 to -1 at $\pi/(2N+2)$, and $f_N^-(\theta)$ is determined from the normalization condition as $f_N^-(\theta) = \pm\sqrt{1 - [f_N^+(\theta)]^2}$. At other regions, $f_N^\pm(\theta)$ are determined by the symmetry requirements. The pulse amplitude $B(t)$ for the generalized UDD is

$$B(t) = \frac{1}{F_N^+(t)} \frac{d}{dt} F_N^-(t) = \sum_{j=1}^N \pi \delta(t - T_j) + B_{\text{extra}}(t) \quad (72)$$

which is a superposition of the instantaneous UDD pulses and an extra component $B_{\text{extra}}(t)$ being arbitrary but subject to the symmetry requirements. The demand of δ -pulses in the generalized UDD is consistent with the previous finding in Ref. [56] that the effect of an instantaneous π -pulse on the evolution of a qubit coupled to a bath cannot be exactly reproduced by a pulse with a finite magnitude. An example of the scaled modulation functions and the corresponding magnetic field for the generalized third-order UDD control are shown in Fig. 2. Notice that due to the variable transformation from θ to t , the magnetic field $B(t)$ does not have the symmetries as the scaled modulation functions $f_N^\pm(\theta)$. For example, $B(t)$ is not periodic and the pulse at different time has different width.

Obviously, the same argument holds for DD against pure dephasing just by changing the rotation axis.

2.3.7 UDD with pulses of finite amplitude

In realistic experiments, the pulses have finite durations and amplitudes, which introduces additional errors. There is a no-go theorem that states that instantaneous π -pulses cannot be approximated by pulses of finite

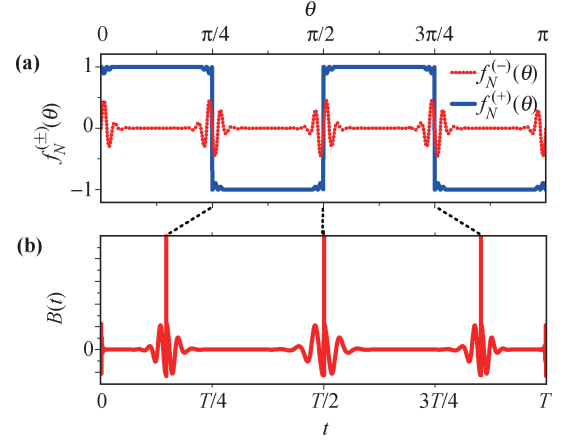


Fig. 2 An example of (a) the scaled modulation functions $f_N^\pm(\theta)$ for the generalized third-order UDD control and (b) the corresponding magnetic field $B(t)$. The dashed lines indicate the correspondence between the sudden jumps of the modulation function $f_N^+(\theta)$ in (a) and the sharp spikes as instantaneous π -pulses in (b). Reproduced from Ref. [50], Copyright © 2008 the American Physical Society.

amplitude and of short duration τ_p with accuracy higher than the order $O(\tau_p)$ without perturbing the bath evolution [56, 57]. However, as we have discussed above, the symmetric requirements of $f_N(\theta)$ automatically guarantee the performance of UDD. Uhrig and Pasini showed that by appropriately designing the pulses, the qubit-bath Hamiltonian describing pure dephasing can be transformed into the form [58]

$$\hat{H} = \hat{C} + \tilde{F}_N(t) \hat{\sigma}_z \otimes \hat{Z} + O(\tau_p^M) \quad (73)$$

with the modulation function taking values from $\{-1, 0, 1\}$. The scaled modulation function $\tilde{f}_N(\theta) \equiv \tilde{F}_N(T \sin^2(\theta/2))$ is designed to have the symmetries required in the previous proof and therefore can be expanded by odd harmonics of $\sin[(N+1)\theta]$. Thus, the decoherence is suppressed up to the order $O(T^{N+1}) + O(\tau_p^M)$. This sequence can also suppress longitudinal relaxation [50, 58]. An arbitrary order M of pulse shaping can be achieved by a recursive scheme based on concatenation [59].

2.4 Comparison of decoupling efficiencies of UDD and CDD

We consider a DD sequence of N pulses, with a total evolution time T and a minimum pulse interval τ . In CDD, the decoupling order is $n \sim \log_2 N$ and $\tau = T/2^n$. In UDD, the decoupling order is N and $\tau \sim T/N^2$. To be specific, our discussion is based on the pure dephasing model. The situation for the general decoherence model is similar. We compare the efficiencies of UDD and CDD in the following scenarios:

Case I. The total evolution time T is fixed. The decoupling precision (defined as the effective coupling under the DD control relative to the original one) in UDD was

derived as [52]:

$$\epsilon_{\text{UDD}} \sim \|H\|^N T^N / N! \quad (74)$$

In CDD, it scales with the time and the decoupling order as [31, 34, 44]:

$$\epsilon_{\text{CDD}} \sim \left(\|H\| T / \sqrt{N} \right)^n = (\|H\| T)^n / 2^{n^2/2} \quad (75)$$

Note that here we have not considered the subtleties in distinguishing the norms of the bath Hamilton and the qubit–bath interaction [31, 34, 44, 52]. Thus, with T fixed, it increases the decoupling order, and hence the number of pulses always increases the decoupling precision. An arbitrarily high decoupling precision can be achieved simply by choosing a sufficiently high order of DD (and correspondingly, a sufficiently small pulse interval τ). In the high-fidelity regime (T is small), the decoupling precision of UDD scales with the number of pulses is much faster than that of CDD. However, if we compare the efficiency of UDD and CDD of the same decoupling order n , i.e., the n th-order UDD (containing n pulses) and the n th-order CDD (containing 2^n pulses), CDD has a much higher decoupling precision than UDD does ($T^n / 2^{n^2/2} \ll T^n / n!$ for large n), since the minimum pulse interval $\tau = T / 2^n$ in CDD is much smaller than that in UDD ($\tau \sim T / n^2$). For the same reason (namely, reduction of τ), to achieve a given order of precision, CDD indeed requires by far less than the seemingly exponential cost.

Case II. The minimum pulse interval τ is fixed, which is a frequently encountered restriction in realistic experiments. In this situation, increasing the order of DD leads to two competing effects [52, 60]. First, the qubit–bath coupling is eliminated to a higher order, which tends to increase the decoupling precision. Second, the total evolution time T increases, and the bath has more time to inflict qubit decoherence. Competition between these two effects leads to the existence of an optimal decoupling order, beyond which further increasing the order of DD does not improve the decoupling precision any longer. For a given minimum pulse interval τ , the optimal order of UDD is [52]

$$n_{\text{opt,UDD}} \sim 1 / (\|H\| \tau) \quad (76)$$

and that of CDD is [34, 43, 44]

$$n_{\text{opt,CDD}} \sim -\log_2 (\|H\| \tau) \quad (77)$$

Note that the total number of pulses is of the same order in the optimal order UDD and CDD. UDD has a much higher optimal level than CDD for a small minimum pulse interval, and therefore the highest decoupling precision that can be achieved by UDD is much higher than that by CDD.

2.5 Experimental progresses

UDD was first realized in experiments by Biercuk *et al.* in an array of ~ 1000 Be^+ ions in a Penning ion trap [40, 61, 62] with noises mimicked by artificially introduced random modulation of the control fields. The qubit states were realized using a ground-state electron-spin-flip transition. Coherent qubit operations were achieved through a quasi-optical microwave system. UDD was compared with CPMG in the “low fidelity” regime for various classical noise spectra. The data show that UDD dramatically outperforms CPMG for Ohmic noise [power spectrum $S(\omega) \propto \omega$] with a sharp cutoff, while for the ambient magnetic field fluctuations whose power spectrum $S(\omega) \propto 1/\omega^4$ has a soft cutoff, UDD performs similarly to CPMG over the entire range of accessible noise intensities, consistent with the theoretical predictions [38, 53].

The first experimental realization of UDD against realistic noises was achieved by Du *et al.* in a solid-state system, namely, irradiated malonic acid single crystals. The spins of the radicals in the crystals created by irradiation form an ensemble of qubits. The nuclear spins, in samples with relatively low concentrations of radicals, constitute the quantum bath, which can be considered as finite for the time-scales involved in the experiment and therefore has a finite noise spectrum. The pulsed electron paramagnetic resonance was used to demonstrate the performance of UDD for preserving electron spin coherence at temperatures from 50 K to room temperature [41]. Using a seven-pulse UDD sequence, the electron spin coherence time was prolonged from 0.04 μs to about 30 μs . The experimental data from different samples under various conditions demonstrate that UDD performs better than CPMG in fighting against noises from nuclear spins. The good agreement between the experiment and calculations based on microscopic theories [16, 17] enables the authors to identify the relevant electron spin decoherence mechanisms as the electron–nuclear contact hyperfine interaction and the electron–electron dipolar interaction.

3 New developments

3.1 CUDD: Concatenation of UDD

CDD can eliminate all the qubit–bath couplings (including pure dephasing and population relaxation) up to an arbitrary order N at the cost of exponentially increasing number (of the order 4^N) of controlling pulses. In contrast, UDD sequence uses the least number (i.e., N) of controlling pulses to eliminate either pure dephasing or population relaxation (but not both) to the desired order N . Based on a combination of CDD and UDD, a new DD

sequence (named CUDD) was proposed [63] to suppress both the pure dephasing and the population relaxation to order N with a much less (of the order $N2^N$) number of pulses. The essential idea of CUDD is to use the N th-order UDD sequence (instead of the free evolution) as the building block of CDD sequence.

The propagator $\hat{U}_{N\text{-UDD}}(T)$ for the qubit-bath evolution driven by the general Hamiltonian Eq. (11) under N th-order UDD sequence of π rotation around the z axis is

$$\begin{aligned}\hat{U}_{N\text{-UDD}} &= e^{-i[\hat{C}' + (-1)^N \hat{D}](T - T_N)} \dots \\ &\cdot e^{-i(\hat{C}' - \hat{D})(T_2 - T_1)} e^{-i(\hat{C}' + \hat{D})T_1} \\ &= e^{-i\hat{H}_{\text{eff}}(T)T + O(T^{N+1})}\end{aligned}\quad (78)$$

[see Eq. (58)], where $\{T_j\}$ are given by Eq. (19), and $\hat{H}_{\text{eff}}(T) = \hat{C}_{\text{eff}}(T) + \hat{\sigma}_z \otimes \hat{Z}_{\text{eff}}(T)$ is a pure dephasing Hamiltonian. The pure dephasing can be eliminated by embedding $\hat{U}_{N\text{-UDD}}$ into the structure $[\hat{\sigma}_x(\dots)\hat{\sigma}_x](\dots)$. The propagator for the m th-order concatenation of $\hat{U}_{N\text{-UDD}}$ is

$$\hat{U}_{N\text{-UDD}}^{(m)} = \hat{\sigma}_x \hat{U}_{N\text{-UDD}}^{(m-1)} \hat{\sigma}_x \hat{U}_{N\text{-UDD}}^{(m-1)} \quad (79)$$

with $\hat{U}_{N\text{-UDD}}^{(0)} = \hat{U}_{N\text{-UDD}}$. In the CUDD scheme, $\hat{U}_{N\text{-UDD}}^{(N)}$ eliminates both the pure dephasing and population relaxation up to the N th order with $O(N2^N)$ pulses.

3.2 Near optimal DD by nesting UDD

Recently, West *et al.* proposed a near optimal DD [64] obtained by nesting UDD sequences, dubbed quadratic DD (QDD), to protect qubits against general noises. The inner N th-order UDD eliminates population relaxation, and the outer N th-order UDD eliminates the pure dephasing, so that both pure dephasing and population relaxation are eliminated up to the N th order of the evolution time. Using $\hat{U}_{N\text{-UDD}}^{(Z)}(\tau)$ to denote the qubit-bath propagator driven by the general Hamiltonian Eq. (11) under the N th-order UDD sequence of π rotation around the z axis, the propagator of the (N, M) th-order near optimal DD,

$$\begin{aligned}\hat{U}_N^{(M)} &= \hat{U}_{N\text{-UDD}}^{(Z)}(T - T_M) \hat{\sigma}_x \hat{U}_{N\text{-UDD}}^{(Z)}(T_M - T_{M-1}) \dots \\ &\cdot \hat{\sigma}_x \hat{U}_{N\text{-UDD}}^{(Z)}(T_2 - T_1) \hat{\sigma}_x \hat{U}_{N\text{-UDD}}^{(Z)}(T_1)\end{aligned}\quad (80)$$

is obtained from Eq. (57) by replacing the free propagator $\hat{U}_0(t)$ by $\hat{U}_{N\text{-UDD}}^{(Z)}(t)$, where $\{T_j\}$ are given by Eq. (19) with N replaced with M . Thus, $\hat{U}_N^{(N)}$ eliminates both the pure dephasing and the population relaxation up to the N th order using $O(N^2)$ pulses. Numerical simulation shows that for a fixed number of pulses, this DD sequence outperforms CDD and CUDD by exponential saving of the number of the pulses and it is nearly optimal for small N , differing from the optimal solutions by

no more than two pulses.

A proof of the QDD was attempted in Ref. [55] with the argument that after the inner level of UDD control, the resulting effective Hamiltonian is time-dependent, and the outer level of UDD control applies to time-dependent Hamiltonians. The effective Hamiltonian under the inner level of UDD control, as defined in Ref. [55], however, is only piecewise analytical. It can be shown by some counter examples [65] that for a general piecewise analytical Hamiltonian taken as resulting from certain inner level of control, it is not guaranteed that the outer level of decoupling can be realized to the desired order. Thus, it remains an open question to us why the nested UDD control works.

3.3 Protecting multi-qubit states by UDD

Mukhtar *et al.* recently showed [66] that by applying a sequence of unitary operations

$$\hat{P}_\psi = 2|\psi\rangle\langle\psi| - I \quad (81)$$

on the multilevel quantum system according to the timing of UDD, the initial quantum state $|\psi\rangle$ is protected to the order of $O(T^{N+1})$. This operation was also given in Ref. [51].

Obviously, we have $\hat{P}_\psi^\dagger = \hat{P}_\psi$. We define the operators

$$\hat{C} = (\hat{H} + \hat{P}_\psi \hat{H} \hat{P}_\psi) / 2 \quad (82a)$$

$$\hat{Z} = (\hat{H} - \hat{P}_\psi \hat{H} \hat{P}_\psi) / 2 \quad (82b)$$

Then, the system-bath Hamiltonian is separated into two parts

$$\hat{H} = \hat{C} + \hat{Z} \quad (83)$$

where \hat{C} commutes with the operator \hat{P}_ψ , while \hat{Z} anticommutes with \hat{P}_ψ , i.e.,

$$\hat{P}_\psi \hat{C} \hat{P}_\psi = \hat{C} \quad (84a)$$

$$\hat{P}_\psi \hat{Z} \hat{P}_\psi = -\hat{Z} \quad (84b)$$

By applying a sequence of N operations \hat{P}_ψ according to the timing of UDD, the system-bath propagator reads

$$\begin{aligned}\hat{U}_N &= \hat{P}_\psi^N e^{-i(\hat{C} + \hat{Z})(T - T_N)} \hat{P}_\psi e^{-i(\hat{C} + \hat{Z})(T_N - T_{N-1})} \dots \\ &\cdot \hat{P}_\psi e^{-i(\hat{C} + \hat{Z})(T_2 - T_1)} \hat{P}_\psi e^{-i(\hat{C} + \hat{Z})T_1}\end{aligned}\quad (85)$$

Note that a final \hat{P}_ψ pulse is required for odd N . Similar to the procedure in the proof of the universality of UDD, we rewrite the propagator as

$$\hat{U}_N = e^{-i\hat{C}T} \hat{\mathcal{T}} e^{-i \int_0^T F_N(t) \hat{Z}_I(t) dt} \quad (86)$$

where $\hat{Z}_I(t) \equiv e^{i\hat{C}t} \hat{Z} e^{-i\hat{C}t}$ anticommutes with \hat{P}_ψ . We separate \hat{U}_N into two parts

$$\hat{U}_N \equiv \hat{U}_N^{(\text{even})} + \hat{U}_N^{(\text{odd})} \quad (87)$$

where

$$\hat{U}_N^{(\text{even})} = e^{-i\hat{C}T} \sum_{k=0}^{\infty} (-i)^{2k} \hat{\Delta}_{2k} \quad (88a)$$

$$\hat{U}_N^{(\text{odd})} = e^{-i\hat{C}T} \sum_{k=0}^{\infty} (-i)^{2k} \hat{\Delta}_{2k+1} \quad (88b)$$

with

$$\hat{\Delta}_n \equiv \int_0^T F_N(t_n) dt_n \int_0^{t_n} F_N(t_{n-1}) dt_{n-1} \cdots \int_0^{t_2} F_N(t_1) dt_1 \hat{Z}_I(t_n) \hat{Z}_I(t_{n-1}) \cdots \hat{Z}_I(t_1) \quad (89)$$

Obviously, $\hat{U}_N^{(\text{even})}$ commutes with \hat{P}_ψ , since it contains even powers of \hat{Z} . Following the same arguments in the proof of UDD for qubit dephasing, we conclude that $\hat{U}_N^{(\text{odd})} = O(T^{N+1})$. Thus, $\hat{P}_\psi \hat{U}_N = \hat{U}_N \hat{P}_\psi + O(T^{N+1})$, which immediately indicate that the expectation value of \hat{P}_ψ , and hence, the quantum state $|\psi\rangle$ are preserved up to $O(T^{N+1})$.

4 Conclusions and perspectives

In summary, we have given a review of recent progresses in protecting qubit coherence by the dynamical decoupling schemes. The DD techniques originate from the magnetic resonance spectroscopy. The developments for quantum information technologies can in turn advance the high-precision magnetic resonance spectroscopy. For example, UDD has recently been applied in magnetic resonance imaging of tumors in animals [67]. The extension of the spin coherence by DD may have important applications in nano scale or even atomic scale magnetometry [68].

Remarkably, experiments have demonstrated the DD method as a particularly promising scheme for protecting quantum coherence in quantum computing. As compared to the quantum error correction schemes, the DD requires no auxiliary qubits and can be integrated naturally with the quantum gates without extra hardware overhead. However, the DD approach has a shortcoming in that it works only for slow baths or for non-Markovian noises, in the sense that the characteristic separation time of the DD sequence is required to be shorter than or at least comparable to the inverse of the characteristic width of the noise spectrum. The quantum error correction scheme has no such requirements. In dealing with errors in quantum computing due to spontaneous emission, combination of DD and quantum error correction was proposed [69]. It is conceivable that in future quantum computing, the non-Markovian noises can be decoupled by DD and the remaining Markovian noises can be coped with by quantum error correction. In general, for a multi-qubit system coupled to both Markovian and non-Markovian noises, a combination of the

two paradigmatic error-countering methods provides a complete picture for scalable quantum computing [44].

In the present research of DD, mostly, the pulses are assumed instantaneous with only a few exceptions. Two important issues are under intensive research, and some remarkable results have emerged recently [31, 42, 44, 52, 59]. One is how to extend the DD to implement high-fidelity quantum gates or hybrid DD with quantum gates. Can some ideas be borrowed from DD for realizing dynamical control resilient to noises? Such an issue was previously addressed in simulation of quantum processors with DD approaches [70]. Recently, encouraging progresses have been made toward hybridization of quantum gates and DD [42–44]. Another issue is how to design a quantum gate (such as a qubit flip, which is required in DD) optimally in the presence of environmental noises. Various optimization schemes have been invented for suppressing/minimizing the noise effect to a certain order [71–74]. Ref. [59] has established a systematic method to achieve an arbitrary order of precision based on iterative construction of finite-amplitude pulses. It is of interest to ask whether and how the pulse shaping for quantum gates with an arbitrary order of precision can be systematically constructed without iteration, with the development from CDD to UDD being an inspiring example.

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