PHYS 5130 Problem Set 7 Solution

1.

Solution: In the lectures, the following results have been derived for $3D$ Fermi gas.

$$E_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{\frac{2}{3}}. \tag{1}$$

Moreover, one could define the following quantities in terms of Fermi energy

$$T_F = \frac{E_F}{k} \tag{2}$$

$$= \frac{\hbar^2}{2mk}\left(3\pi^2 n\right)^{\frac{2}{3}}. \tag{3}$$

$$K_F = \frac{\sqrt{2mE_F}}{\hbar} \tag{4}$$

$$= (3\pi^2 n)^{\frac{1}{3}} \tag{5}$$

(a) Given $n = 2.65 \times 10^{22}\,\mathrm{cm}^{-3} = 2.65 \times 10^{28}\,\mathrm{m}^{-3}$, and $m_e = 9.109\,383\,701\,5 \times 10^{-31}\,\mathrm{kg}$. One could obtain Fermi energy E_F and Fermi temperature T_F by direct substitution into the formulae.

One could then obtain

$$E_F = 5.1929 \times 10^{-19}\,\mathrm{J} = 3.24\,\mathrm{eV}, \tag{6}$$

and

$$T_F = 3.76 \times 10^4\,\mathrm{K}. \tag{7}$$

(b) Average separation is given by $\left(\frac{V}{N}\right)^{\frac{1}{3}} = 3.354\,\text{Å}$.

$$T_0 = \frac{h^2}{2\pi mk}\left(\frac{N}{V}\right)^{\frac{2}{3}} \tag{8}$$

$$= 4.9384 \times 10^4\,\mathrm{K} \tag{9}$$

T_F and T_0 are of the same order.

(c) Using $n = 1 \times 10^{44}\,\mathrm{m}^{-3}$, one obtains

$$E_F = 6.8455 \times 10^{-12}\,\mathrm{J} = 4.2726 \times 10^7\,\mathrm{eV}, \tag{10}$$

$$T_F = 4.958 \times 10^{11}\,\mathrm{K} \tag{11}$$

and

$$K_F = 1.4359 \times 10^{15}\,\mathrm{m}^{-1} = 1.4359 \times 10^5\,\text{Å}^{-1}. \tag{12}$$

(d) Here, $n = 1 \times 10^{16}\,\mathrm{cm}^{-3} = 1 \times 10^{22}\,\mathrm{m}^{-3}$. So one obtains

$$E_F = 2.718 \times 10^{-23}\,\mathrm{J} = 1.69 \times 10^{-4}\,\mathrm{eV} \tag{13}$$

and

$$T_F = 1.96\,\mathrm{K}. \tag{14}$$

2.

Solution:

(a) As starting point, one could make use of the density of state $g(k)$ for particle in a box. Then, one could compute $g^{<}(k)$.

$$g^{<}(k) = \int_0^k g(k)dk \tag{15}$$
$$= \frac{4\pi}{8}\left(\frac{L}{\pi}\right)^3 \int_0^k k^2 dk \tag{16}$$
$$= \frac{L^3}{6\pi^2}k^3. \tag{17}$$

Making use of $\epsilon = c\hbar k$, one could then obtain $g^{<}(\epsilon)$

$$g^{<}(\epsilon) = \frac{L^3}{6\pi^2}\left(\frac{\epsilon}{c\hbar}\right)^3. \tag{18}$$

Therefore,

$$g(\epsilon) = \frac{\mathrm{d}g^{<}(\epsilon)}{\mathrm{d}\epsilon} \tag{19}$$
$$= \frac{L^3}{2\pi^2 c^3\hbar^3}\epsilon^2 \tag{20}$$
$$= \frac{V}{2\pi^2 c^3\hbar^3}\epsilon^2. \tag{21}$$

Taking into consideration the spin degeneracy factor $g_s = 2$, one obtains

$$g(\epsilon) = \frac{V}{\pi^2 c^3\hbar^3}\int_0^\infty \epsilon^2 \tag{22}$$

(b) The equation for total number of particles is given by

$$N = \int_0^\infty g(\epsilon) f_{FD}(\epsilon) d\epsilon. \tag{23}$$

At $T = 0$, the integral can be written as

$$N = \int_0^{E_F} g(\epsilon) d\epsilon \tag{24}$$
$$= \frac{V}{\pi^2 c^3\hbar^3}\int_0^{E_F} \epsilon^2 d\epsilon \tag{25}$$
$$= \frac{VE_F^3}{3\pi^2 c^3\hbar^3}. \tag{26}$$

One then obtains $E_F = c\hbar\left(3\pi^2 n\right)^{\frac{1}{3}}$.
Total energy of the system is given by

$$E = \int_0^\infty g(\epsilon) f_{FD}(\epsilon)\epsilon d\epsilon. \tag{27}$$

At $T = 0$,

$$E = \int_0^\infty g(\epsilon) f_{FD}(\epsilon)\epsilon d\epsilon. \tag{28}$$
$$= \frac{V}{\pi^2 c^3\hbar^3}\int_0^{E_F} \epsilon^3 d\epsilon \tag{29}$$
$$= \frac{VE_F^4}{4\pi^2 c^3\hbar^3} \tag{30}$$
$$= \frac{3}{4}NE_F \tag{31}$$

So, energy per particle is given by

$$\frac{E}{N} = \frac{3}{4}E_F. \tag{32}$$

(c) In the lectures, the following result has been derived for 3D ideal Fermi gas,

$$pV = kT\sum_i g_i \ln\left(1+e^{-\beta(\epsilon_i-\mu)}\right). \tag{33}$$

In this continuum case,

$$\begin{aligned}
pV &= kT\int_0^\infty g(\epsilon)\ln\left(1+e^{-\beta(\epsilon-\mu)}\right)d\epsilon & (34)\\
&= \frac{kTV}{\pi^2c^3\hbar^3}\int_0^\infty \epsilon^2\ln\left(1+e^{-\beta(\epsilon-\mu)}\right)d\epsilon & (35)\\
&= \frac{kTV}{3\pi^2c^3\hbar^3}\int_0^\infty \ln\left(1+e^{-\beta(\epsilon-\mu)}\right)d\epsilon^3 & (36)\\
&= \frac{kTV}{3\pi^2c^3\hbar^3}\left(\epsilon^3\ln\left(1+e^{-\beta(\epsilon-\mu)}\right)\Big|_0^\infty+\beta\int_0^\infty\frac{\epsilon^3e^{-\beta(\epsilon-\mu)}}{1+e^{-\beta(\epsilon-\mu)}}d\epsilon\right) & (37)\\
&= \frac{V}{3\pi^2c^3\hbar^3}\int_0^\infty\frac{\epsilon^3e^{-\beta(\epsilon-\mu)}}{1+e^{-\beta(\epsilon-\mu)}}d\epsilon & (38)\\
&= \frac{1}{3}\int_0^\infty\frac{g(\epsilon)\epsilon}{e^{\beta(\epsilon-\mu)}+1}d\epsilon & (39)\\
&= \frac{1}{3}\int_0^\infty g(\epsilon)\epsilon f_{FD}(\epsilon)d\epsilon & (40)\\
&= \frac{1}{3}E & (41)
\end{aligned}$$

(d) Using results from part (b), (c), one obtains

$$\begin{aligned}
pV &= \frac{1}{4}NE_F & (42)\\
p &= \frac{1}{4}nE_F & (43)\\
&= \left(\frac{3^{\frac{1}{3}}\pi^{\frac{2}{3}}}{4}\right)c\hbar n^{\frac{4}{3}} & (44)
\end{aligned}$$

3.

Solution:

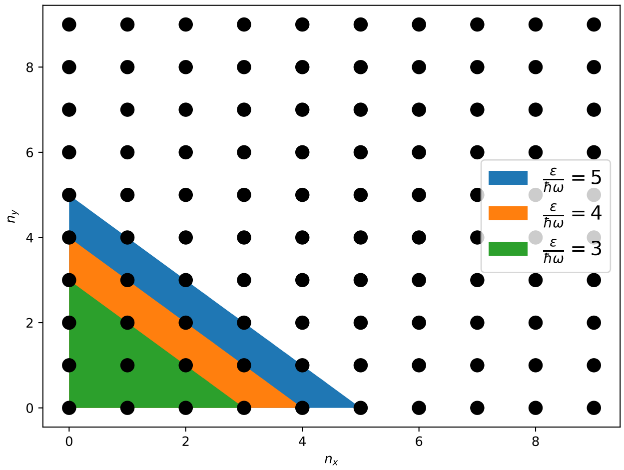
(a) Rather than considering the density of state $g(\epsilon)$ directly, one could first consider the number of states with energy less than or equal ϵ, denoted by $g^<(\epsilon)$.

After some consideration, one could see that $g^<(\epsilon)$ corresponds to a right angled pyramid in the n space, with height $\frac{\epsilon}{\hbar\omega}$, base area $\frac{1}{2}\left(\frac{\epsilon}{\hbar\omega}\right)^2$. So $g^<(\epsilon) = \frac{\epsilon^3}{6(\hbar\omega)^3}$. (Ignoring any possible spin degeneracy factor.)

Therefore,

$$\begin{aligned}
g(\epsilon) &= \frac{\mathrm{d}g^<(\epsilon)}{\mathrm{d}\epsilon} & (45)\\
&= \frac{\epsilon^2}{2(\hbar\omega)^3}. & (46)
\end{aligned}$$

For illustration, one could consider the 2D case, the figure below shows the 2D n space, with each point indicating one state, which occupies an area of unit 1. The shaded regions indicates right angled (isosceles) triangles of different lengths for the sides, as area increases, the relative contribution of states on the border decreases, and the number of states within the triangle can be approximated by its area, which can be identified with $g^<_{2D}(\epsilon)$. Therefore, in 2D case, $g^<_{2D}(\epsilon) = \frac{1}{2}\left(\frac{\epsilon}{\hbar\omega}\right)^2$. The situation is similar in 3D case, except one has to consider the volume of a right angled pyramid instead.



(b) At T_c, $\mu = 0$. Therefore, one obtains

$$N = \int_0^\infty g(\epsilon) f_{BE}(\epsilon) d\epsilon \tag{47}$$

$$= \frac{1}{2\left(\hbar\omega\right)^3} \int_0^\infty \frac{\epsilon^2}{e^{\frac{\epsilon}{kT_c}} - 1} d\epsilon. \tag{48}$$

Letting $x = \frac{\epsilon}{\hbar\omega}$, the above expression could be rewritten as

$$N = \frac{(kT_C)^3}{2(\hbar\omega)^3} \int_0^\infty \frac{x^2}{e^x - 1} dx \tag{49}$$

$$= \frac{(kT_C)^3}{2(\hbar\omega)^3} \Gamma(3)\zeta(3) \tag{50}$$

$$= \frac{(kT_C)^3}{2(\hbar\omega)^3} \Gamma(3)\zeta(3) \tag{51}$$

$$= \left(\frac{kT_C}{\hbar\omega}\right)^3 \zeta(3) \tag{52}$$

Then, one obtains

$$T_C = \frac{\hbar\omega}{k} \left(\frac{N}{\zeta(3)}\right)^{\frac{1}{3}} \tag{53}$$

(c)

$$N_0 = N - N_{excited} \tag{54}$$

$$= N - \frac{1}{2\left(\hbar\omega\right)^3} \int_0^\infty \frac{\epsilon^2}{e^{\frac{\epsilon}{kT}} - 1} d\epsilon \tag{55}$$

$$= N - \left(\frac{kT}{\hbar\omega}\right)^3 \zeta(3) \tag{56}$$

$$= N - \left(\frac{T}{T_C}\right)^3 \left(\frac{kT_C}{\hbar\omega}\right)^3 \zeta(3) \tag{57}$$

$$= N - N\left(\frac{T}{T_C}\right)^3 \tag{58}$$

$$= N\left(1 - \left(\frac{T}{T_C}\right)^3\right) \tag{59}$$

4.

Solution: The potential function is given by

$$U = \begin{cases} \infty & r < r_c \\ -c_6 r^{-6} & r > r_c \end{cases}. \tag{60}$$

For the given potential, one could compute B_2,

$$\begin{aligned} B_2 &= -2\pi \int_0^\infty \left(e^{-\frac{U}{kT}} - 1\right) r^2 dr && (61) \\ &= -2\pi \int_0^{r_c} \left(e^{-\frac{U}{kT}} - 1\right) r^2 dr - 2\pi \int_{r_c}^\infty \left(e^{-\frac{U}{kT}} - 1\right) r^2 dr && (62) \\ &= 2\pi \int_0^{r_c} r^2 dr - 2\pi \int_{r_c}^\infty \left(e^{\frac{c_6}{r^6 kT}} - 1\right) r^2 dr && (63) \\ &= \frac{2\pi}{3} r_c^3 - 2\pi \int_{r_c}^\infty \left(e^{\frac{c_6}{r^6 kT}} - 1\right) r^2 dr && (64) \\ &\approx \frac{2\pi}{3} r_c^3 - 2\pi \int_{r_c}^\infty \frac{c_6}{r^4 kT} dr && (65) \\ &= \frac{2\pi}{3} r_c^3 - \frac{2\pi c_6}{3 r_c^3} \frac{1}{kT}. && (66) \end{aligned}$$

Comparison with the form of B_2 obtained from van der Waals equation leads to the identification of $\frac{b}{N_A}$ with $\frac{2\pi}{3} r_c^3$ and $\frac{a}{N_A^2}$ with $\frac{2\pi c_6}{3 r_c^3}$.

5.

Solution:

(a) At the coexistence line, the two phases share the same temperature and pressure, so one can write

$$\frac{8T_R}{3\left(v_g - \frac{1}{3}\right)} - \frac{3}{v_g^2} = \frac{8T_R}{3\left(v_g - \frac{1}{3}\right)} - \frac{3}{v_g^2}. \tag{67}$$

Therefore, one could write the expression in terms of T_R, given by

$$T_R = \frac{1}{8}\left(\frac{v_l + v_g}{v_g^2 v_l^2}\right)(3v_g - 1)(3v_l - 1). \tag{68}$$

Introducing the following two variables, $s = v_g + v_l$, $\Delta v = v_g - v_l$, the expression could be rewritten as

$$T_R = 2s\left(\left(\frac{3}{2}s - 1\right)^2 - \frac{9}{4}\Delta v^2\right)(s^2 - \Delta v^2)^{-2} \tag{69}$$

To explore the behavior around critical point, one could perform Taylor expansion around $v_l = 1, v_g = 1$, so $s = 2, \Delta v = 0$. After some calculation, to the lowest nonvanishing order, one obtains

$$\begin{aligned} T_R(2 + \delta s, \delta v) &= T_R(2, 0) - \frac{3}{16}\delta s^2 - \frac{1}{16}\delta v^2 && (70) \\ \delta T_R &= -\frac{3}{16}\delta s^2 - \frac{1}{16}\delta v^2 && (71) \end{aligned}$$

As $\delta v_g > 0, \delta v_l < 0$, so δs^2 term can be neglected. One then obtains $\beta = \frac{1}{2}$.

(b) Fixing $T_R = 1$, one obtains

$$p_R = \frac{8}{3\left(v_R - \frac{1}{3}\right)} - \frac{3}{v_R^2} \tag{72}$$

To extract the behavior of p near the critical point, with $T_R = 1$, one could consider the taylor expansion of $p_R(v_R)$ at $v_R = 1$, one then obtains

$$\begin{aligned} p_R(1 + \delta v_R) &= p_R(1) + \left.\frac{\mathrm{d}p_R}{\mathrm{d}v_R}\right|_1 \delta v_R + \frac{1}{2}\left.\frac{\mathrm{d}^2 p_R}{\mathrm{d}v_R^2}\right|_1 \delta v_R^2 + \dots && (73) \\ \delta p_R &= \left.\frac{\mathrm{d}p_R}{\mathrm{d}v_R}\right|_1 \delta v_R + \frac{1}{2}\left.\frac{\mathrm{d}^2 p_R}{\mathrm{d}v_R^2}\right|_1 \delta v_R^2 + \dots && (74) \end{aligned}$$

Then, to extract the critical exponent, one has to evaluate the different order of derivatives of p_R.

$$\frac{\mathrm{d}p_R}{\mathrm{d}v_R} = -\frac{8}{3\left(v_R - \frac{1}{3}\right)^2} + \frac{6}{v_R^3}. \tag{75}$$

As $\left.\frac{\mathrm{d}p_R}{\mathrm{d}v_R}\right|_1 = 0$, one has to go to higher order.

$$\frac{\mathrm{d}^2 p_R}{\mathrm{d}v_R^2} = \frac{16}{3\left(v_R - \frac{1}{3}\right)^3} - \frac{18}{v_R^4}. \tag{76}$$

Similarly, $\left.\frac{\mathrm{d}^2 p_R}{\mathrm{d}v_R^2}\right|_1 = 0$.

$$\frac{\mathrm{d}^3 p_R}{\mathrm{d}v_R^3} = -\frac{48}{3\left(v_R - \frac{1}{3}\right)^4} + \frac{72}{v_R^5}. \tag{77}$$

As $\left.\frac{\mathrm{d}^3 p_R}{\mathrm{d}v_R^3}\right|_1 = -9$, to the lowest order, one gets

$$\delta p_R = -\frac{3}{2}\delta v_R^3. \tag{78}$$

So $\delta = 3$.