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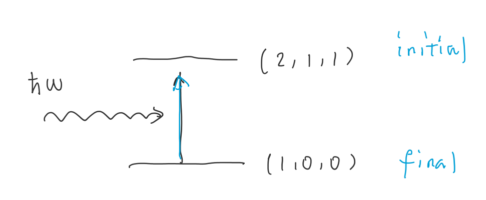
SQ 20.

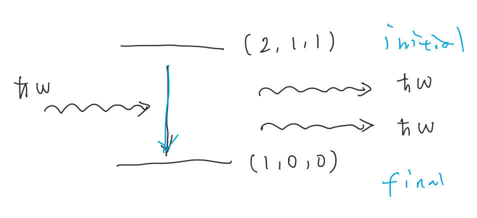
Refering to SQ 19, we consider a 2-state transition of Hydrogen atom between 2p ($n=2$, $l=1$, $m_l=1$) and 1s ($n=1$, $l=0$, $m_l=0$) state.

There are 3 possible transitions:

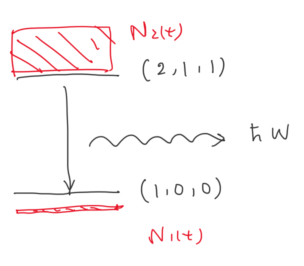
① Stimulated absorption: from (1,0,0) to (2,1,1),



② Stimulated emission: from (2,1,1) to (1,0,0),



③ Spontaneous emission: from (2,1,1) to (1,0,0),



In SQ 19, we have discussed that process ① can be estimated by time-depending perturbation, and the coefficient with final state (2,1,1) can be estimated by the integral:

$$a_{(2,1,1)}(t) \propto \overline{r}_{(2,1,1),(1,0,0)} = \int \psi^*_{(2,1,1)}(\vec{r})\, \vec{r}\, \psi_{(1,0,0)}(\vec{r})\, d^3r$$

In SQ 19, we obatined $\overline{r}_{(2,1,1),(1,0,0)} = \frac{128}{243} a_0 (\hat{x} - i\hat{y})$,

which is a complex vector.

(a) For process ②, we notice that now the final state and intial state are exchanged, i.e. the final state is (1,0,0) and ~~the~~ intial state is (2,1,1).

And the perturbation term is the same:

$$\hat{H}' = -\vec{\mu}_{el} \cdot \vec{\varepsilon}, \quad \text{where} \quad \vec{\mu}_{el} = -e\vec{r}.$$

Therefore, $\overline{r}_{1s,2p} = \int \psi^*_{(1,0,0)}(\vec{r})\, \vec{r}\, \psi_{(2,1,1)}\, d^3r$

$$= \left[\int \psi^*_{(2,1,1)}(\vec{r})\, \underline{\vec{r}}\, \psi_{(1,0,0)}\, \underline{d^3r} \right]^*$$

$$= \vec{r}^{\,*}_{2p,1s} \qquad (\vec{r},\ d^3r \text{ are real})$$

$$= \frac{128}{243} a_0 (\hat{x} + i\hat{y}),$$

which is also a complex vector.

(b) The transition probability $|a_2(t)|^2$ depends on

$$|\overline{r}_{1s,2p}|^2 = \overline{r}_{1s,2p}^{\,*} \cdot \overline{r}_{1s,2p}$$

$$= \left(\frac{128}{243} a_0\right)^2 (\hat{x} - i\hat{y})(\hat{x} + i\hat{y})$$

$$= 2\left(\frac{128}{243} a_0\right)^2$$

$$\approx 0.555\, a_0^2$$

(c) We use the Einstein coefficients A and B to denote the transition probability per atom per unit time:

① Stimulated absorption: $B_{12}\, U(\omega, T)$

② Stimulated emission: $B_{21}\, U(\omega, T)$

③ Spontaneous emission: A_{21}

(1 denotes for (1,0,0)
2 denotes for (2,1,1))

From class notes, we obtained a formula for stimulated absorption ①:

$$B_{21} = \frac{\pi}{3\varepsilon_0 \hbar^2} |\overline{\mu}_{21}|^2, \quad \text{where } \overline{\mu}_{21} = e\,\overline{r}_{21} \text{ is the electric dipole moment.}$$

Similarly, for stimulated emission ②:

$$B_{12} = \frac{\pi}{3\varepsilon_0 \hbar^2} |\overline{\mu}_{12}|^2, \quad \text{where } \overline{\mu}_{12} = e\,\overline{r}_{12}$$

From (a), we know that $\overline{r}_{12} = \overline{r}_{21}^{\,*}$.

(i) $|\overline{\mu}_{12}|^2 = e^2 |\overline{r}_{12}|^2 = e^2 (2) \left(\frac{128}{243}\right)^2 a_0^2 \approx 3.988 \times 10^{-39}\ C^2 m^2$,

where $e = 1.602 \times 10^{-19}\ C$, $a_0 = 5.292 \times 10^{-11}\ m$.

And $|\overline{\mu}_{21}|^2 = e^2 |\overline{r}_{21}|^2 = |\overline{\mu}_{12}|^2$.

Hence we get $B = B_{12} = B_{21} = \frac{\pi}{3\varepsilon_0 \hbar^2} (3.988 \times 10^{-39}\ C^2 m^2)$

(ii) ω (or ω_{21}) is the angular frequency of the incoming photon, i.e.

$$\omega = \omega_{21} = \frac{E_2 - E_1}{\hbar},$$

where $E_1 = -13.6\ eV$ is the energy of 1s state, $E_2 = -2.4\ eV$ is the energy of 2p state.

Put in the values, $\omega \approx 1.550 \times 10^{16}\ s^{-1}$.

(d) Obtained the value of $|\overline{\mu}_{21}|^2$ and ω, we can now compute what is A:

$$A = \frac{\omega^3}{3\pi\varepsilon_0 c^3 \hbar} e^2 |\overline{r}_{21}|^2 = \frac{\omega^3}{3\pi\varepsilon_0 c^3 \hbar} |\overline{\mu}_{21}|^2,$$

where $\omega = 1.550 \times 10^{16}\ s^{-1}$, $|\overline{\mu}_{21}| = 3.998 \times 10^{-39}\ C^2 m^2$,

$h = 6.626 \times 10^{-34}\ Js$, $c = 2.998 \times 10^8\ ms^{-1}$, $\varepsilon_0 = 8.854 \times 10^{-12}\ m^{-3} kg^{-1} s^4 A^2$.

Then we get $A = 6.26 \times 10^8\ s^{-1}$.

And the lifetime τ is $\tau = \frac{1}{A} = 1.60 \times 10^{-9}\ s$.

This lifetime τ is a measurable quantity in experiment. And it is typical of a state that can make a transition downward via electric dipole radiation.

SQ 21.

We study the natural line broadening here. We consider the electric field of the signal for a spontaneous emission in the form:

$$\varepsilon(t) = \begin{cases} 0, & t < 0 \\ \varepsilon_0 e^{-i\omega_{21} t} e^{-t/2\tau}, & t > 0 \end{cases}$$

which naturally satisfies $I(t) \propto \underbrace{N_2(t)}_{\text{(number of atoms in state 2)}} \sim e^{-t/\tau}$.

(a) We do a Fourier transform:

$$\varepsilon(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varepsilon(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \varepsilon_0\, e^{i(\omega - \omega_{21} + \frac{i}{2\tau})t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \varepsilon_0 \frac{1}{i(\omega - \omega_{21} + \frac{i}{2\tau})} \underbrace{\left[e^{i(\omega - \omega_{21} + \frac{i}{2\tau})t} \right]_0^{\infty}}_{-1}$$

(when $t \to \infty$, $e^{i(\omega - \omega_{21})t} e^{-t/2\tau} \to 0$)

$$= \frac{1}{\sqrt{2\pi}} \varepsilon_0 \frac{i}{(\omega - \omega_{21}) + \frac{i}{2\tau}}$$

(b) The signal light intensity

$$I(\omega) \propto |\varepsilon|^2 = \varepsilon^*(t)\, \varepsilon(t)$$

$$= \frac{\varepsilon_0^2}{2\pi} \frac{-i}{(\omega - \omega_{21}) - \frac{i}{2\tau}} \frac{i}{(\omega - \omega_{21}) + \frac{i}{2\tau}}$$

$$= \frac{\varepsilon_0^2}{2\pi} \frac{1}{(\omega - \omega_{21})^2 + \frac{1}{4\tau^2}}$$

(c) It is obvious that when $\omega = \omega_{21}$, $I(\omega)$ reaches the maximum.

($\omega = \omega_{21}$, the denominator reaches minimum and thereby $I(\omega)$ reaches maximum).

The peak $I(\omega_{21}) \propto \frac{\varepsilon_0^2}{2\pi} \frac{1}{\frac{1}{4\tau^2}} = \frac{\varepsilon_0^2}{2\pi} \cdot 4\tau^2$.

Hence we have $\frac{I(\omega)}{I(\omega_{21})} = \frac{1}{4\tau^2} \frac{1}{(\omega - \omega_{21})^2 + \frac{1}{4\tau^2}}$,

where we obtain $I(\omega) = \frac{I(\omega_{21})}{4\tau^2} \frac{1}{(\omega - \omega_{21})^2 + \frac{1}{4\tau^2}} = \frac{I_{peak}}{4\tau^2} \frac{1}{(\omega - \omega_{21})^2 + \frac{1}{4\tau^2}}$.

(d) The half maximum has intensity $\frac{1}{2} I_{peak}$.

Suppose at ω' (we assume $\omega' > \omega_{21}$) we reach the half maximum:

$$I(\omega') = \frac{I_{peak}}{4\tau^2} \frac{1}{(\omega' - \omega_{21})^2 + \frac{1}{4\tau^2}} = \frac{1}{2} I_{peak},$$

$$\Rightarrow \frac{1}{4\tau^2(\omega' - \omega_{21})^2 + 1} = \frac{1}{2}$$

$$4\tau^2 (\omega' - \omega_{21})^2 = 1$$

$$\omega' - \omega_{21} = \pm \frac{1}{2\tau}$$

$$\omega' = \omega_{21} \pm \frac{1}{2\tau}$$

Hence the full width at half maximum FWHM $\Delta\omega = (\omega_{21} + \frac{1}{2\tau}) - (\omega_{21} - \frac{1}{2\tau})$

$$= \frac{1}{\tau}.$$

(e) Since we obtained $\Delta\omega = \frac{1}{\tau}$, we can replace $\frac{1}{4\tau^2}$ by $\underbrace{\left(\frac{\Delta\omega}{2}\right)^2}$,

Lorentz spectral line-shape function

We sketch $I(\omega) = I(\omega_{21}) \left[\frac{\left(\frac{\Delta\omega}{2}\right)^2}{(\omega - \omega_{21})^2 + \left(\frac{\Delta\omega}{2}\right)^2} \right]$ as: