SQ8

Minimal substitution rule for electron $(q=-e)$:

$$H = \frac{\vec{p}^2}{2m} + V(r) \quad \rightarrow \quad H = \frac{(\vec{p}+e\vec{A})^2}{2m} + V(r)$$

$\vec{B} = \vec{\nabla}\times\vec{A}$

$V(r) = -\dfrac{e^2}{4\pi\varepsilon_0 r}$

For $\vec{B} = B\hat{z}$, we can choose a vector potential such that $\vec{B} = \vec{\nabla}\times\vec{A}$

One possible choice is $\vec{A} = \frac{1}{2}Bx\hat{y} - \frac{1}{2}By\hat{x}$:

$$\vec{\nabla}\times\vec{A} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} = B\hat{z}$$

Expand the Hamiltonian by

$$\vec{p} = p_x\hat{x} + p_y\hat{y} + p_z\hat{z}$$

$$\vec{A} = -\frac{1}{2}By\hat{x} + \frac{1}{2}Bx\hat{y}$$

we have

$$H = \frac{1}{2m}\left[(p_x - \frac{e}{2}By)\hat{x} + (p_y + \frac{e}{2}Bx)\hat{y} + p_z\hat{z}\right]^2 + V(r)$$

$$= \frac{1}{2m}\left[(p_x - \frac{e}{2}By)^2 + (p_y + \frac{e}{2}Bx)^2 + p_z^2\right] + V(r)$$

$$= \frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2 - eByp_x + eBxp_y + \frac{e^2}{4}B^2y^2 + \frac{e^2}{4}B^2x^2\right] + V(r)$$

$$= \frac{1}{2m}\left[p^2 + eB(xp_y - yp_x) + \frac{e^2}{4}B^2(x^2+y^2)\right] + V(r)$$

$p^2 = p_x^2 + p_y^2 + p_z^2$

L_z : z-component of angular momentum

SQ8

$$eB(xp_y - yp_x) = eBL_z = e\vec{B}\cdot\vec{L} \qquad (\vec{B} = B\hat{z})$$

$$\because \quad \vec{\mu}_L = -\frac{e}{2m}\vec{L}$$

$$\therefore \quad eB(xp_y - yp_x) = -2m\vec{B}\cdot\vec{\mu}_L$$

The Hamiltonian becomes :

$$H = \frac{p^2}{2m} - \vec{B}\cdot\vec{\mu}_L + \frac{e^2}{8m}B^2(x^2+y^2) + V(r)$$

$$= \frac{p^2}{2m} + V(r) - \vec{B}\cdot\vec{\mu}_L + \frac{e^2}{2m}A^2$$

$$A^2 = A_x^2 + A_y^2 + A_z^2$$
$$= \frac{1}{4}B^2(x^2+y^2)$$

Paramagnetic term showing the interaction energy between magnetic dipole moment and external magnetic field

diamagnetic response which is proportional to A^2

In QM, all these terms become operators.

SQ9

For the labelling scheme $(j_1 = 5/2,\ m_{j_1},\ j_2 = 1,\ m_{j_2})$, we have

$\lvert j_1 = 5/2,\ m_{j_1} = 5/2,\ j_2 = 1,\ m_{j_2} = 1\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -1/2,\ j_2 = 1,\ m_{j_2} = 1\rangle$
$\lvert j_1 = 5/2,\ m_{j_1} = 5/2,\ j_2 = 1,\ m_{j_2} = 0\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -1/2,\ j_2 = 1,\ m_{j_2} = 0\rangle$
$\lvert j_1 = 5/2,\ m_{j_1} = 5/2,\ j_2 = 1,\ m_{j_2} = -1\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -1/2,\ j_2 = 1,\ m_{j_2} = -1\rangle$
$\lvert j_1 = 5/2,\ m_{j_1} = 3/2,\ j_2 = 1,\ m_{j_2} = 1\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -3/2,\ j_2 = 1,\ m_{j_2} = 1\rangle$
$\lvert j_1 = 5/2,\ m_{j_1} = 3/2,\ j_2 = 1,\ m_{j_2} = 0\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -3/2,\ j_2 = 1,\ m_{j_2} = 0\rangle$
$\lvert j_1 = 5/2,\ m_{j_1} = 3/2,\ j_2 = 1,\ m_{j_2} = -1\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -3/2,\ j_2 = 1,\ m_{j_2} = -1\rangle$
$\lvert j_1 = 5/2,\ m_{j_1} = 1/2,\ j_2 = 1,\ m_{j_2} = 1\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -5/2,\ j_2 = 1,\ m_{j_2} = 1\rangle$
$\lvert j_1 = 5/2,\ m_{j_1} = 1/2,\ j_2 = 1,\ m_{j_2} = 0\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -5/2,\ j_2 = 1,\ m_{j_2} = 0\rangle$
$\lvert j_1 = 5/2,\ m_{j_1} = 1/2,\ j_2 = 1,\ m_{j_2} = -1\rangle$	$\lvert j_1 = 5/2,\ m_{j_1} = -5/2,\ j_2 = 1,\ m_{j_2} = -1\rangle$

We get $(2\cdot\frac{5}{2}+1)(2\cdot 1+1) = 18$ states in this labelling

SQ9

For the labelling scheme $(j_1 = 5/2,\ j_2 = 1,\ j,\ m_j)$:

$|j_1=5/2,\ j_2=1,\ j=7/2,\ m_j=7/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=7/2,\ m_j=5/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=7/2,\ m_j=3/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=7/2,\ m_j=1/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=7/2,\ m_j=-1/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=7/2,\ m_j=-3/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=7/2,\ m_j=-5/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=7/2,\ m_j=-7/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=5/2,\ m_j=5/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=5/2,\ m_j=3/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=5/2,\ m_j=1/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=5/2,\ m_j=-1/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=5/2,\ m_j=-3/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=5/2,\ m_j=-5/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=3/2,\ m_j=3/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=3/2,\ m_j=1/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=3/2,\ m_j=-1/2\rangle$

$|j_1=5/2,\ j_2=1,\ j=3/2,\ m_j=-3/2\rangle$

We also have 18 states in this labelling

SQ9

Number of states for (j_1, m_{j_1}) : $2j_1 + 1$

Number of states for (j_2, m_{j_2}) : $2j_2 + 1$

$\therefore$ Number of states for $(j_1, m_{j_1}, j_2, m_{j_2}) = (2j_1+1)(2j_2+1)$

Total angular momentum quantum number ranges from $j_1 - j_2$ to $j_1 + j_2$ in steps of 1 if $j_1 > j_2$. For each j inside this range, we have $(2j+1)$ values of m_j. So the total number of states in this label:

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = 2\sum_{j=j_1-j_2}^{j_1+j_2} j + \sum_{j=j_1-j_2}^{j_1+j_2} 1$$

$$= 2\,\frac{2j_2+1}{2}\left[2(j_1-j_2) + 2j_2\right] + 2j_2+1$$

(The first term can be calaulated by sum of arithmetic series while the second term just counts the number of j between j_1-j_2 and j_1+j_2)

$$= (2j_2+1)(2j_1-2j_2+2j_2+1)$$

$$= (2j_2+1)(2j_1+1)$$

$\therefore$ Number of states in the two labels are the same.

SQ10

$$\psi_{200} = \frac{1}{\sqrt{2\pi a}} \frac{1}{2a}\left(1-\frac{r}{2a}\right) e^{-r/2a} \qquad \psi_{210} = \frac{1}{\sqrt{2\pi a}} \frac{1}{4a^2} r e^{-r/2a} \cos\theta$$

$$\psi_{211} = -\frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta \, e^{i\phi} \qquad \psi_{21-1} = \frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta \, e^{-i\phi}$$

To set up the 4x4 matrix for degenerate perturbation theory, we need to calculate the matrix elements of H' : $\langle \psi_{2lm} | \hat{H}' | \psi_{2l'm'} \rangle$

$$\langle \psi_{200} | H' | \psi_{200} \rangle \propto \int_0^{\pi} \cos\theta \sin\theta \, d\theta = 0$$

$$\langle \psi_{211} | H' | \psi_{211} \rangle \propto \int_0^{\pi} \sin^2\theta \cos\theta \sin\theta \, d\theta = 0$$

$$\langle \psi_{210} | H' | \psi_{210} \rangle \propto \int_0^{\pi} \cos^2\theta \cos\theta \sin\theta \, d\theta = 0$$

$$\langle \psi_{21-1} | H' | \psi_{21-1} \rangle \propto \int_0^{\pi} \sin^2\theta \cos\theta \sin\theta \, d\theta = 0$$

$$\langle \psi_{200} | H' | \psi_{211} \rangle \propto \int_0^{2\pi} e^{i\phi} d\phi = 0$$

$$\langle \psi_{200} | H' | \psi_{21-1} \rangle \propto \int_0^{2\pi} e^{-i\phi} d\phi = 0$$

$$\langle \psi_{211} | H' | \psi_{210} \rangle \propto \int_0^{2\pi} e^{-i\phi} d\phi = 0$$

$$\langle \psi_{211} | H' | \psi_{21-1} \rangle \propto \int_0^{2\pi} e^{-2i\phi} d\phi = 0$$

$$\langle \psi_{210} | H' | \psi_{21-1} \rangle \propto \int_0^{2\pi} e^{-i\phi} d\phi = 0$$

The transposes of above matrix elements are also zero because H' is hermitian ($\langle m | H' | n \rangle = \langle n | H' | m \rangle^*$) The only non-vanishing matrix elements are

$\langle \psi_{200} | H' | \psi_{210} \rangle$ and $\langle \psi_{210} | H' | \psi_{200} \rangle$

S910

This selection rule holds even if the electric field is applied in the $\hat{x}$ or $\hat{y}$ direction because we can always choose a new coordinate system with z-axis aligning with the electric field and the physics is the same.

Since only $\langle \psi_{200} | H' | \psi_{210} \rangle$ and $\langle \psi_{210} | H' | \psi_{200} \rangle$ are non-zero, the original 4x4 matrix reduces effectively to the following 2x2 problem:

$$\begin{vmatrix} E_2^{(0)} - E & \langle \psi_{200} | H' | \psi_{210} \rangle \\ \langle \psi_{210} | H' | \psi_{200} \rangle & E_2^{(0)} - E \end{vmatrix} = 0$$

ψ_{211} and ψ_{21-1} are unaffected by H' and they still have the same unperturbed eigenenergy $E_2^{(0)} \sim -\frac{13.6}{4}$ eV

(of course, you can do the 4x4 matrix problem. You will find that two of the four eigenvalues are just $E_2^{(0)}$ and their corresponding eigenvectors are something like $(0\ 0\ 1\ 0)^T$ and $(0\ 0\ 0\ 1)^T$ which correspond to ψ_{211} and ψ_{21-1} respectively)

Now, we calculate $\langle \psi_{200} | H' | \psi_{210} \rangle$

$$\langle \psi_{200} | H' | \psi_{210} \rangle$$

$$= e\mathcal{E} \frac{1}{\sqrt{2\pi a}} \frac{1}{2a} \frac{1}{\sqrt{2\pi a}} \frac{1}{4a^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \left(1 - \frac{r}{2a}\right) e^{-r/2a}\, r e^{-r/2a} \cos\theta\, \underbrace{(r\cos\theta)}_{z}\; r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{e\mathcal{E}}{16\pi a^4}\, 2\pi \underbrace{\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta}_{2/3} \int_0^{\infty} \left(1 - \frac{r}{2a}\right) e^{-r/a}\, r^4\, dr$$

SQ10

$$\langle \psi_{200} | H' | \psi_{210} \rangle$$

$$= \frac{e\varepsilon}{12a^4} \left[\underbrace{\int_0^\infty r^4 e^{-r/a} dr}_{24a^5} - \frac{1}{2a} \underbrace{\int_0^\infty r^5 e^{-r/a} dr}_{120a^6} \right]$$

$$= -3e\varepsilon a$$

The 2x2 matrix :

$$\begin{vmatrix} E_2^{(0)} - E & -3e\varepsilon a \\ -3e\varepsilon a & E_2^{(0)} - E \end{vmatrix} = 0$$

$$E = E_2^{(0)} + 3e\varepsilon a \quad \text{or} \quad E_2^{(0)} - 3e\varepsilon a$$

Corresponding eigenstate : $\frac{1}{\sqrt{2}}(\psi_{200} - \psi_{210})$

Corresponding eigenstate : $\frac{1}{\sqrt{2}}(\psi_{200} + \psi_{210})$