SQ1

(a) In spectroscopy : $\bar{\nu} \equiv \frac{1}{\lambda}$ ← wavelength

↑ wavenumber

For transition from $n_2 \rightarrow n_1$ (assumes $n_2 > n_1$)

$$E_{n_2} - E_{n_1} = \frac{me^4}{8\varepsilon_0^2 h^2}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

Frequency of light emitted during the transition :

$$E_{n_2} - E_{n_1} = hf$$

$$f = \frac{me^4}{8\varepsilon_0^2 h^3}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

Using the relation $c = f\lambda$,

$$\lambda = \frac{c}{f} = \frac{8\varepsilon_0^2 h^3 c}{me^4}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)^{-1}$$

$$\bar{\nu} = \frac{me^4}{8\varepsilon_0^2 c h^3}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

Calculating the prefactor :

$$R_H = \frac{me^4}{8\varepsilon_0^2 c h^3} \approx 109737 \text{ cm}^{-1}$$

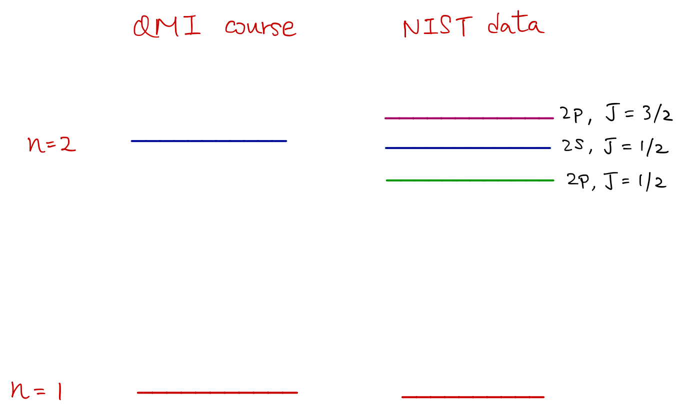
It is the limit when $n_2 \rightarrow \infty$ and $n_1 = 1$

NIST data of Hydrogen Atom

Configuration	Term	J	Level (cm^{-1})	Ref.
1s	^2S	1/2	0.0000	MK00a
2p	$^2P^\circ$	1/2	82258.9191	MK00a
		3/2	82259.2850	MK00a
2s	^2S	1/2	82258.9544	MK00a
3p	$^2P^\circ$	1/2	97492.2112	MK00a
		3/2	97492.3196	MK00a
3s	^2S	1/2	97492.2217	MK00a
3d	^2D	3/2	97492.3195	MK00a
		5/2	97492.3556	MK00a
4p	$^2P^\circ$	1/2	102823.8486	MK00a
		3/2	102823.8943	MK00a
4s	^2S	1/2	102823.8530	MK00a
4d	^2D	3/2	102823.8942	MK00a
		5/2	102823.9095	MK00a
4f	$^2F^\circ$	5/2	102823.9095	MK00a
		7/2	102823.9171	MK00a
5p	$^2P^\circ$	1/2	105291.6287	MK00a
		3/2	105291.6521	MK00a
5s	^2S	1/2	105291.6309	MK00a
5d	^2D	3/2	105291.6520	MK00a
		5/2	105291.6599	MK00a
5f	$^2F^\circ$	5/2	105291.6598	MK00a
		7/2	105291.6637	MK00a
5g	^2G	7/2	105291.6637	MK00a
		9/2	105291.6661	MK00a
H	*Limit*		**109678.7717**	MK00a

Energy levels of H atom not only depend on the quantum number n

More interestingly, energies can be different even we have same n and l (e.g. 2p) It is the effect of spin-orbit coupling and you will learn it later in this course



The value is smaller than our results in part (a)

To take into account of the fact that H-atom is a two-body problem, we should replace m in Eq (2) by the reduced mass μ

$$\mu = m_{proton} m_e / (m_{proton} + m_e) = 0.999456\, m_e$$

The corresponding value of

$$R_H = \frac{\mu e^4}{8 \varepsilon_0^2 h^3 c} \approx 109677 \text{ cm}^{-1}$$

SQ2

Relativistic kinetic energy :

$$T = \sqrt{m^2c^4 + p^2c^2} - mc^2$$

$$= mc^2 \sqrt{1 + \left(\frac{p}{mc}\right)^2} - mc^2$$

Taylor expansion of $\sqrt{1+x}$

$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$

Here, we just substitute $x = \left(\frac{p}{mc}\right)^2$

For small relativistic correction, $\frac{p}{mc} << 1$

$$T = mc^2\left[1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2 - \frac{1}{8}\left(\frac{p}{mc}\right)^4 + \cdots\right] - mc^2$$

$$\approx \underbrace{\frac{p^2}{2m}}_{\hat{H}_0} - \underbrace{\frac{p^4}{8m^3c^2}}_{\hat{H}'}$$

In Bohr's model, the $n=1$ orbit is a circular orbit with radius $r = \underbrace{a}_{\text{Bohr radius}}$

$$\frac{mv^2}{a} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{a^2}$$

$$v^2 = \frac{e^2}{4\pi\varepsilon_0 a m} \qquad \left(a = \frac{4\pi\varepsilon_0\hbar^2}{me^2}\right)$$

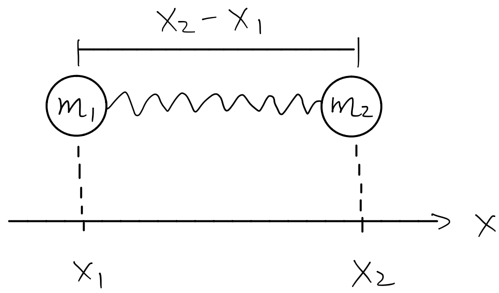
$$v = \frac{e^2}{4\pi\varepsilon_0\hbar}$$

$$\frac{v}{c} = \frac{e^2}{4\pi\varepsilon_0 c\hbar} = \underbrace{\alpha}_{\text{Fine-structure constant}} \approx \frac{1}{137}$$

The speed of electron is near 1% of c, really not slow !

SQ3

The extension of spring from its natural length is $x_2 - x_1 - r_0$. Then by Newton's 2nd law,



$$\begin{cases} m_1 \dfrac{d^2 x_1}{dt^2} = k(x_2 - x_1 - r_0) & \cdots (1) \\ m_2 \dfrac{d^2 x_2}{dt^2} = -k(x_2 - x_1 - r_0) & \cdots (2) \end{cases}$$

Eq (1) + Eq (2) gives

$$m_1 \frac{d^2 x_1}{dt^2} + m_2 \frac{d^2 x_2}{dt^2} = 0$$

$$\frac{d^2}{dt^2}\left(m_1 x_1 + m_2 x_2 \right) = 0$$

$$(m_1 + m_2) \frac{d^2}{dt^2}\left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \right) = 0$$

$$M \frac{d^2 X_{CM}}{dt^2} = 0$$

$M = m_1 + m_2 =$ total mass

$X_{CM} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$=$ coordinate of center-of-mass

This equation shows that the center-of-mass is moving freely in time with constant momentum. The "0" in the RHS also indicates that there is no external force.

SQ3

$m_2 \cdot$ Eq(1) $-$ $m_1 \cdot$ Eq(2) gives

$$m_1 m_2 \frac{d^2}{dt^2}(x_1 - x_2) = k(m_1 + m_2)(x_2 - x_1 - r_0)$$

Let $x = x_2 - x_1$

$$m_1 m_2 \frac{d^2 x}{dt^2} = -k(m_1 + m_2)(x - r_0)$$

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2 x}{dt^2} = -k(x - r_0)$$

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2}{dt^2}(x - r_0) = -k(x - r_0)$$

r_0 is a constant $\Rightarrow \frac{dr_0}{dt} = 0$

$$\mu \frac{d^2 r}{dt^2} + kr = 0$$

where $\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the system

$r = x - r_0 = x_2 - x_1 - r_0$ is the extension of spring from its natural length

This equation shows that the relative motion of the system is a simple harmonic oscillator.

$$w = \sqrt{\frac{k}{\mu}} \qquad f = \frac{w}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}} \qquad \bar{\nu} = \frac{1}{\lambda} = \frac{1}{2\pi c}\sqrt{\frac{k}{\mu}}$$