Notes on

LINEAR ALGEBRA

with a few remarks on

LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

1. Systems of linear equations

Suppose we are given a system of m linear equations in n unknowns $x_1, \ldots, x_n$:

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
\tag{1}
$$

To write this in a more compact form we introduce a matrix and two vectors,

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}, \qquad
\mathbf{b} = \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix} \qquad \text{and} \qquad
\mathbf{x} = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{pmatrix},
$$

so that (1) becomes $A\mathbf{x} = \mathbf{b}$. In the next section we turn to the problem of determining whether or not the system has any solutions and, if it does, of finding all of them. Before that, however, we make some general comments on the nature of solutions. Notice how similar these are to the results we saw earlier for linear second-order differential equations; we will point out in Section 7 below their similarity to results for linear systems of differential equations.

The homogeneous problem. Suppose first that the system (1) is *homogeneous*, that is, that the right hand side is zero, or equivalently that $b_1 = b_2 = \cdots = b_m = 0$ or $\mathbf{b} = \mathbf{0}$:

$$
A\mathbf{x} = \mathbf{0}.
\tag{2}
$$

Suppose further that we have found, by some method, two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the equations. Then for any constants c and d, $\mathbf{x} = c\mathbf{x}_1 + d\mathbf{x}_2$ is also a solution, since

$$
A\mathbf{x} = A(c\mathbf{x}_1 + d\mathbf{x}_2) = cA\mathbf{x}_1 + dA\mathbf{x}_2 = c \cdot \mathbf{0} + d \cdot \mathbf{0} = \mathbf{0}.
$$

The argument extends to any number of solutions, and we have the

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Principle 1: Superposition. If $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_k$ are all solutions of (2), and c_1, c_2, $\ldots$, c_k are constants, then

$$
\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k
\tag{3}
$$

is also a solution of (2).

The name of this principle comes from the fact that (3) is called a *linear combination* or *linear superposition* of the solutions $\mathbf{x}_1, \ldots, \mathbf{x}_k$. We will see later (see Principle 3 (iii) on page 8) that there is a special value of k such that (a) we can find a set of solutions $\mathbf{x}_1, \ldots, \mathbf{x}_k$ with the property that *every* solution of (1) can be built as a linear combination of these solutions, and (b) k different solutions are really needed for this to be true.

Notice also that **the homogeneous system always has at least one solution, the zero solution $\mathbf{x} = \mathbf{0}$**, since $A\mathbf{0} = \mathbf{0}$.

The inhomogeneous problem. Consider now the case in which the system (1) is *inhomogeneous*, that is, $\mathbf{b}$ is arbitrary. Suppose again that we are given two solutions, which we will now call $\mathbf{x}$ and $\mathbf{X}$. Then $\mathbf{x}_h = \mathbf{x} - \mathbf{X}$ is a solution of the *homogeneous* system, since

$$A\mathbf{x}_h = A(\mathbf{x} - \mathbf{X}) = A\mathbf{x} - A\mathbf{X} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

This means that if we know *one* solution of our equations, X, then every other solution is obtained by $\mathbf{x} = \mathbf{X} + \mathbf{x}_h$:

Principle 2: Inhomogeneous linear equations. Every solution $\mathbf{x}$ of the system of inhomogeneous equations (1) is of the form $\mathbf{x} = \mathbf{X} + \mathbf{x}_h$, where $\mathbf{X}$ is some *particular* solution of the system, and $\mathbf{x}_h$ is a solution of the corresponding homogeneous system.

In particular, if $x_1, \ldots, \mathbf{x}_k$ are the special solutions of the homogeneous equation referred to above and in Principle 3 (iii), then

$$\mathbf{x} = \mathbf{X} + c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k. \tag{4}$$

2. Row reduction and row-echelon form

The key technique that we will use for solving linear equations, and also for investigating general properties of the solutions, is the *reduction of a matrix to row-echelon form* or to *reduced row-echelon form* by the use of *elementary row operations*, a procedure often called *row reduction* or *Gaussian elimination*. Symbolically, if A is a matrix, we have

$$A \xrightarrow[\text{operations}]{\text{elementary row}} R$$

where R is in row-echelon or reduced row-echelon form. What does this all mean?

Row-echelon form: The matrix R is in row-echelon form (REF) if it satisfies three conditions:

(i) All nonzero rows (that is, rows with at least one nonzero entry) are above any zero rows (rows with all zeros).

(ii) The first nonzero entry in any nonzero row is a 1. This entry is called a *pivot*.

(iii) Each pivot lies to the right of the pivot in the row above it.

Here is a typical matrix in row-echelon form:

$$R = \begin{pmatrix} 0 & \mathbf{1} & 3 & -2 & 3 & 5 & 0 & 12 \\ 0 & 0 & 0 & \mathbf{1} & -2 & 0 & -15 & 5 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

The pivots are the entries, all with value 1, shown in boldface. Boyce and DiPrima reduce matrices to this form when they do Gaussian elimination—see, e.g., Examples 1 and 2 in Section 7.3—but they don't use the terminology.

Warning: The text by Spence, Insel and Friedberg, used in Math 250, has a different definition of row-echelon form: the pivots are the first nonzero entries in the nonzero rows, but they are not required to have value 1. Unfortunately, both definitions are in common use.

Reduced row-echelon form: It is sometimes convenient to carry the reduction further, and bring the matrix into *reduced row-echelon form* (RREF). This form satisfies conditions (i)–(iii) above, and also

(iv) All matrix entries above a pivot are zero.

When the matrix R of (5) is put into reduced row-echelon form, it becomes

$$R' = \begin{pmatrix} 0 & \mathbf{1} & 3 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & \mathbf{1} & -2 & 0 & 0 & 65 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 28 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

One advantage, more theoretical than practical, is that the RREF of a matrix A is unique—whatever sequence of row operations is used to go from A to R, with R in RREF, the resulting R will be the same. Boyce and DiPrima do not used the reduced row-echelon form.

Elementary row operations: There are three elementary row operations on matrices:

    R1. Interchange of two rows.

    R2. Multiplication of a row by a nonzero scalar.

    R3. Addition of a multiple of one row to another row.

By using these operations repeatedly we can bring any matrix into row echelon form. The procedure is illustrated on the next page, and there are also worked out examples in Boyce and DiPrima.

Rank: If you do the row operations in different ways you can arrive at different REF matrices R from the same starting matrix A. However, all the REF matrices you find will

Example 1: Row reduction

Here we carry out the reduction of a 3×4 matrix first to row-echelon, and then to reduced row-echelon, form. We indicate the row operations used by a simple notation: $\mathbf{r}_i$ denotes the ith row of the matrix, and the row operations are denoted by $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$ (interchange rows i and j), $\mathbf{r}_i \to c\,\mathbf{r}_i$ (multiply row i by the scalar c), and $\mathbf{r}_i \to \mathbf{r}_i + c\,\mathbf{r}_j$ (add c times row j to row i). Notice that in the first step we *must* switch the first row with another: because the first column is not identically zero, the first pivot must be in the upper left corner, and we need a nonzero entry there to get started.

$$
\begin{pmatrix} 0 & -3 & -1 & 1 \\ 1 & 2 & 3 & 0 \\ 2 & 2 & 5 & -3 \end{pmatrix}
\xrightarrow{\ \mathbf{r}_1 \leftrightarrow \mathbf{r}_2\ }
\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -1 & 1 \\ 2 & 2 & 5 & -3 \end{pmatrix}
$$

$$
\xrightarrow{\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\,\mathbf{r}_1\ }
\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -1 & 1 \\ 0 & -2 & -1 & -3 \end{pmatrix}
$$

$$
\xrightarrow{\ \mathbf{r}_2 \to -(1/3)\,\mathbf{r}_2\ }
\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & -2 & -1 & -3 \end{pmatrix}
$$

$$
\xrightarrow{\ \mathbf{r}3 \to \mathbf{r}3 + 2\,\mathbf{r}2\ }
\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & -1/3 & -11/3 \end{pmatrix}
$$

$$
\xrightarrow{\ r_3 \to -3\,\mathbf{r}_3\ }
\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 1 & 11 \end{pmatrix}
$$

This completes the reduction of A to row-echelon form. If we like, we can continue the process and reach reduced row-echelon form:

$$
\xrightarrow[\ \mathbf{r}_2 \to \mathbf{r}_2 - (1/3)\,\mathbf{r}_3\ ]{\ \mathbf{r}_1 \to \mathbf{r}_1 - 3\,\mathbf{r}_3\ }
\begin{pmatrix} 1 & 2 & 0 & -33 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 11 \end{pmatrix}
$$

$$
\xrightarrow{\ \mathbf{r}_1 \to \mathbf{r}_1 - 2\,\mathbf{r}_2\ }
\begin{pmatrix} 1 & 0 & 0 & -25 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 11 \end{pmatrix}
$$

The extra steps for the reduction to reduced row-echelon form could also have been done at the same time as the earlier steps; for example, at the fourth step above, when we did $\mathbf{r}_3 \to \mathbf{r}_1 + 2\,\mathbf{r}_2$, we could also have done $\mathbf{r}_1 \to \mathbf{r}_1 - 2\,\mathbf{r}_2$ to leave the pivot as the only nonzero entry in column 2.

have the same number of nonzero rows. The number of nonzero rows in R is called the *rank* of A, and written $\text{rank}(A)$ (it is also the rank of R, since R is already in REF).

In the rest of these notes it is assumed that the reader knows what the row-echelon form and reduced row-echelon forms are and, given a matrix A, knows how to reduce it to row-echelon form and/or reduced row-echelon form. **Please review the definitions above and the procedure outlined in Example 1 to be sure that these concepts are clear.**

3. Solving systems of linear equations

Suppose now that we are given the system of linear equations (1) and want to determine whether or not it has any solutions and, if so, to find them all. The idea is to solve (1) by doing elementary operations on the equations, corresponding to the elementary row operations on matrices: interchange two equations, multiply an equation by a nonzero constant, or add a multiple of one equation to another. What is important is that these operations do not change the set of solutions of the equations, so that we can reduce the equations to simpler form, solve the simple equation, and know that we have found the all solutions of the original equations, but no extraneous ones. Moreover, instead of working with the equations, we can work with the *augmented matrix*:

$$(A \,|\, \mathbf{b}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & \vdots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{pmatrix}.$$

(It's not necessary to write the vertical bars here, but they remind us that the last column plays a special role.) Simplifying the original set of equations is equivalent to reducing the augmented matrix to REF or RREF. Once this is done, we can easily find the solutions explicitly, if there are any. Equally important, just by looking at the REF or RREF we can determine whether solutions exist and, if so, many of their properties. We will write this symbolically as

$$(A \,|\, \mathbf{b}) \xrightarrow{\text{elementary row operations}} (R \,|\, \mathbf{e})$$

The entire new augmented matrix $(R \,|\, \mathbf{e})$ is supposed to be in REF; this means that we have also reduced A to the REF matrix R.

Example 2: Suppose we want to solve the equations

$$\begin{aligned} -3x_2 - x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 2x_2 + 5x_3 &= -3 \end{aligned} \tag{6}$$

The augmented matrix is the one we studied in the example in Example 1, so we already know a row-echelon form for it:

$$(A \,|\, \mathbf{b}) = \begin{pmatrix} 0 & -3 & -1 & 1 \\ 1 & 2 & 3 & 0 \\ 2 & 2 & 5 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{1} & 2 & 3 & 0 \\ 0 & \mathbf{1} & 1/3 & -1/3 \\ 0 & 0 & \mathbf{1} & 11 \end{pmatrix} = (R \,|\, \mathbf{e}).$$

The REF corresponds to the equations

$$\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 0 \\
x_2 + (1/3)x_3 &= -1/3 \\
x_3 &= 11
\end{aligned} \tag{7}$$

These may be solved by the process of "back substitution": solve first for x_3, substitute that value into the previous equation and solve for x_2, then substitute both values into the first equation to find x_1:

$$x_3 = 11, \quad x_2 = -1/2 - (1/3)x_3 = -4; \quad x_1 = -2x_2 - 3x_3 = -25, \quad \text{so} \quad \mathbf{x} = \begin{pmatrix} -25 \\ -4 \\ 11 \end{pmatrix}.$$

Notice that in this example the equations have a solution, and it is unique.

    If we had used the reduced row-echelon form (which we also found in Example 1) we would have found the solution more quickly:

$$(A \,|\, \mathbf{b}) = \begin{pmatrix} 0 & -3 & -1 & 1 \\ 1 & 2 & 3 & 0 \\ 2 & 2 & 5 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{1} & 0 & 0 & -25 \\ 0 & \mathbf{1} & 0 & -4 \\ 0 & 0 & \mathbf{1} & 11 \end{pmatrix} = (R' \,|\, \mathbf{e}').$$

The work we did in the first method, doing back substitution, is equivalent to the extra steps used to find the RREF in Example 1. Technically the first procedure—solving the system by finding the REF, then using back substitution—is called *Gaussian elimination*, and the second procedure is called *Gauss-Jordan elimination*, but we will not make this distinction, referring to either simply as Gaussian elimination.

    In the next examples we will omit the step of row reduction and start with a matrix in reduced row-echelon form. We choose RREF because that makes the calculations somewhat simpler, but none of our conclusions would be different if we had used REF and back substitution.

Example 3: Suppose that the RREF form of the augmented matrix is

$$(R \,|\, \mathbf{e}) = \begin{pmatrix} \mathbf{1} & 2 & 0 & 1 & | & 5 \\ 0 & 0 & \mathbf{1} & 3 & | & 2 \\ 0 & 0 & 0 & 0 & | & \mathbf{1} \end{pmatrix}.$$

The last equation here is $0 = 1$, which clearly has no solutions: it expresses a contradiction. This is the signal that our original equations have no solutions. Notice that one way to say what has happened here is that the rank of R, which is 2, is less than the rank of $(R \,|\, \mathbf{e})$, which is three. In general, we will have no solution precisely if $\operatorname{rank}(R) < \operatorname{rank}(R \,|\, \mathbf{e})$.

Example 4: Suppose that the RREF form of the augmented matrix is

$$(R \,|\, \mathbf{e}) = \begin{pmatrix} 0 & \mathbf{1} & 2 & 0 & 1 & | & 5 \\ 0 & 0 & 0 & \mathbf{1} & 3 & | & 2 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Now the idea is to solve for the variables x_2 and x_4, the variables for the columns containing pivots, in terms of the other variables, which are treated as parameters. To remind us that we are treating these variables as parameters, we will give them new names: $\alpha = x_1$, $\beta = x_3$, and $\gamma = x_5$. then our solution is

$$x_1 = \alpha, \quad x_2 = 5 - 2\beta - \gamma, \quad x_3 = \beta, \quad x_4 = 2 - 3\gamma, \quad x_5 = \gamma.$$

In vector form,

$$\mathbf{x} = \begin{pmatrix} \alpha \\ 5 - 2\beta - \gamma \\ \beta \\ 2 - 3\gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \\ 1 \end{pmatrix}. \tag{8}$$

Here we have three parameters, one for each column of R which does not contain a pivot. There are $n = 5$ unknowns and $r = \operatorname{rank}(R) = 2$ pivots, and subtracting these numbers indeed gives $n - r = 3$ free parameters.

The pattern here is quite general. A solution will exist if $\operatorname{rank}(R) = \operatorname{rank}(R \,|\, \mathbf{e})$, and it will have the general form

$$\mathbf{x} = \mathbf{X} + c_1 \mathbf{x}_1 + c_2 \mathbf{x}_x + \cdots + c_k \mathbf{x}_k.$$

The free parameters $c_1, \ldots c_k$ are just the original unknowns corresponding to the columns without pivots. Since there are $r = \operatorname{rank}(R) = \operatorname{rank}(A)$ pivots there will be $n - r$ free parameters in the solution (that is, $k = n - r$). Since we can choose the parameters freely, we can take $c_1 = c_2 = \cdots = c_k = 0$ and we thus find that $\mathbf{X}$ itself a solution. This is the *particular* solution we discussed in Section 1. If we consider now the homogeneous problem—the same equations, but with $\mathbf{b} = 0$—then we will also have $\mathbf{e} = 0$, and by looking at (8) we can see that we will have $\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_x + \cdots + c_k \mathbf{x}_k$ with the *same* vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$; this means that we have recovered (4).

We summarize in Principle 3 on page 8.

4. The case of n equations in n unknowns

Probably the most common systems of linear equations have the same number of equations as unknowns—say n equations in n unknowns. The coefficient matrix A is then *square*, with n rows and n columns. In this case there is a connection between the questions of whether a solution *exists*, and whether a solution which does exist is *unique*. As we shall see, one of two things may happen. Suppose that the augmented matrix has been reduced to RREF $(R \,|\, \mathbf{e})$.

Case 1: $\operatorname{rank}(A) = n$. Since R is an $n \times n$ matrix in RREF with no zero rows, it must be the identity matrix, so that $(R \,|\, \mathbf{e}) = (I \,|\, \mathbf{e})$. The corresponding equations $x_1 = e_1$, $x_2 = e_2, \ldots, x_n = e_n$ will have a solution $\mathbf{x} = \mathbf{e}$ no matter what $\mathbf{e}$ is, and hence no matter what the original $\mathbf{b}$ was; moreover, the solution is clearly always unique.

Principle 3: Solving linear equations. Suppose that the augmented matrix $(A \,|\, \mathbf{b})$ is reduced to the REF or RREF $(R \,|\, \mathbf{e})$. Then:

(i) If $\mathrm{rank}(R) < \mathrm{rank}(R \,|\, \mathbf{e})$, so that the last nonzero equation is $0 = 1$, then the equations have no solutions. This cannot happen if the system is homogeneous.

(ii) If $\mathrm{rank}(R) = \mathrm{rank}(R \,|\, \mathbf{e})$ then the equations have at least one solution. Write $r = \mathrm{rank}(R) = \mathrm{rank}(A)$; then the solution is unique if $n = r$, i.e., if every column in R has a pivot. Otherwise, the equations have a family of solutions with $k = n - r$ free parameters. The general solution may be written in the form

$$\mathbf{x} = \mathbf{X} + c_1 \mathbf{x}_1 + c_2 \mathbf{x}_x + \cdots + c_k \mathbf{x}_k, \tag{9}$$

where X is a particular solution, $c_1, \ldots, c_k$ are the parameters, and $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are solutions of the homogeneous equations $A\mathbf{x} = \mathbf{0}$. The specific solutions are found by solving the reduced equations for the variables corresponding to the columns with pivots in terms of the other variables, which become the parameters.

(iii) The *homogeneous* system always has at least one solution: $\mathbf{x} = \mathbf{0}$. This is the *trivial* solution. The system has nontrivial solutions if and only if there are columns in R which do not contain pivots, that is, if and only if $r < n$. The general solution of the homogeneous equation is of the form

$$\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_x + \cdots + c_k \mathbf{x}_k, \tag{10}$$

with $k = r - n$.

Case 2: $\mathrm{rank}(A) < n$. In this case, the last row of R is a zero row. This means that for some choices of $\mathbf{b}$, the right hand side of the original equations, the vector $\mathbf{e}$ can have one more nonzero component than there are nonzero rows in R, i.e., that the equations will have no solution for some $\mathbf{b}$. On the other hand, if a solution does exist, then because there is a column without a pivot, our solution method will lead to a solution with at least one free parameter—that is, any solution that does exist will not be unique. We have:

Principle 4: n equations in n unknowns. If A is a square matrix then the system of equations $A\mathbf{x} = \mathbf{b}$ **either** has a unique solution for every $\mathbf{b}$ (Case 1), **or** fails to have a solution for some $\mathbf{b}$, and never has a unique solution (Case 2).

Note, for example, that if we know that for some $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then we must be in Case 1 and we immediately know that it has a solution, and in fact a unique solution, for *every* $\mathbf{b}$. Note also that the homogeneous system $A\mathbf{x} = \mathbf{0}$ can have a nontrivial solution only in Case 2, that is, if and only if $\mathrm{rank}(A) = 0$.

There is another way to distinguish between Case 1 and Case 2 which we will use but not prove: **we are in Case 1, that is, $\mathrm{rank}(A) = n$, only if the determinant of A, $\det(A)$, is not zero**.

Much more can be said in Case 1. Suppose that we are in this case, i.e., that $\operatorname{rank}(A) = n$. Let us define the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ to be the columns of the $n \times n$ identity matrix:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{u}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

We know that the system $A\mathbf{x} = \mathbf{u}_i$ has a unique solution, which we will call $\mathbf{v}_i$, that is, $A\mathbf{v}_i = \mathbf{u}_i$. Now consider a matrix B with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$: $B = (\mathbf{v}_1 \ \mathbf{v}_2 \cdots \ \mathbf{v}_n)$. Because of the definition of matrix multiplication, if we compute AB we just multiply each column of B by the matrix A: thus

$$AB = (A\mathbf{v}_1 \ A\mathbf{v}_2 \ A\mathbf{v}_3 \ \cdots \ A\mathbf{v}_m) = (\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3 \ \cdots \ \mathbf{u}_n) = I.$$

Now we say that an $n \times n$ matrix A is *invertible* if it has an *inverse*: a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$ (A^{-1} must necessarily also be $n \times n$). We want to show that if A falls under Case 1 then it is invertible and the matrix B found above is A^{-1}. To do so we observe that B also falls under Case 1, since if $\mathbf{x}$ is a vector with $B\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = I\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$, so that the equations $B\mathbf{x} = 0$ have a unique solution. But then by the argument above there is a matrix C with $BC = I$, and then we have $A = AI = ABC = IC = C$, so $BA = BC = I$ and with $AB = I$ this shows that $B = A^{-1}$. It is also clear that if A is invertible then it must fall under Case 1, since the equations $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$.

These ideas also tell us how to compute A^{-1}. First, how do we find $\mathbf{v}_i$? We do Gaussian elimination on the augmented matrix $(A \,|\, \mathbf{u}_i)$, and $\mathbf{v}_i$, the solution, will just be the last column of the result, that is, the row reduction will be $(A \,|\, \mathbf{u}_i) \to (I \,|\, \mathbf{v}_i)$. Doing all these different problems to find all the $\mathbf{v}_i$ is a terrible duplication of effort, however, so we do them all at once:

$$(A | \mathbf{u}_1 \ \mathbf{u}_2 \ \cdots \mathbf{u}_n) \to (I | \mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \mathbf{v}_n) \quad \text{or equivalently} \quad (A \,|\, I) \to (I \,|\, A^{-1}).$$

This method of computing A^{-1} is illustrated in Boyce and DiPrima, Section 7.2, Example 2.

We can conclude that if A is a square matrix then any one of the following conditions is enough to guarantee that we are in Case 1, and hence that in fact all the conditions hold:

C1: The system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

C2: Whenever the system $A\mathbf{x} = \mathbf{b}$ has a solution, the solution is unique.

C3. The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

C4: $\operatorname{rank}(A) = n$

C5: A has an inverse matrix A^{-1} satisfying $AA^{-1} = A^{-1}A = I$.

C6: The reduced row-echelon form of A is the identity matrix I.

C7: The determinant of A is not zero.

5. Linear independence of vectors

The concept of linear independence of vectors is discussed in Boyce and DiPrima, Section 7.3 (pages 377-379), but we will add a few remarks. Suppose we are given k vectors $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_k$, these might be either row or column vectors, but they are all one or the other, and they all have the same number of components. We then ask the question: can any one of these vectors be expressed as a linear combination of the remaining ones? If so, the vectors are *linearly dependent*, if not, they are *linearly independent*.

Example 5: (a) The vectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\mathbf{x}_3 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ are linearly *dependent*, since $\mathbf{x}_3$ can be expressed as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$: $\mathbf{x}_3 = 3\mathbf{x}_1 - 2\mathbf{x}_2$,

(b) The vectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, are linearly *independent*. For example, we cannot write $\mathbf{x}_1 = a\mathbf{x}_2 + b\mathbf{x}_3$ no matter how we choose a and b, since $a\mathbf{x}_2 + b\mathbf{x}_3 = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}$ has first component 0, and $\mathbf{x}_1$ has first component 1.

(c) The vectors $\mathbf{x}_1 = (\begin{matrix} 1 & 5 & -3 & 2 \end{matrix})$, $\mathbf{x}_2 = (\begin{matrix} 0 & 0 & 0 & 0 \end{matrix})$, and $\mathbf{x}_3 = (\begin{matrix} 7 & -1 & 2 & 0 \end{matrix})$ are linearly *dependent*, since $\mathbf{x}_2 = 0\mathbf{x}_1 + 0\mathbf{x}_3$. Clearly, any set of vectors in which one vector is $\mathbf{0}$ must be linearly dependent, by the same reasoning.

There is another way to describe linear dependence: the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_k$ are linearly dependent if there exist scalars $c_1, \ldots, c_k$, **not all zero**, such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}. \tag{11}$$

The restriction that not all the c_i be zero is important, since we could always make (11) true by taking $c_1 = c_2 = \cdots = c_k = 0$. This new definition of linear dependence is the same as our original definition. For if the vectors are linearly dependent according to our first definition then one of them, say $\mathbf{x}_1$, can be expressed as a linear combination of the others: $\mathbf{x}_1 = d_2\mathbf{x}_2 + d_3\mathbf{x}_3 + \cdots + d_k\mathbf{x}_k$; but then

$$\mathbf{x}_1 - d_2\mathbf{x}_2 - d_3\mathbf{x}_3 - \cdots - d_k\mathbf{x}_k = \mathbf{0},$$

which shows that (11) holds with the coefficients c_i not all zero (since $c_1 = 1$). Conversely, if (11) holds with some coefficient not zero—say, $c_1 \neq 0$—then we can solve the equation for $\mathbf{x}_1$, expressing it as a linear combination of the others:

$$\mathbf{x}_1 = -\left(\frac{c_2}{c_1}\right)\mathbf{x}_2 - \cdots - \left(\frac{c_k}{c_1}\right)\mathbf{x}_k,$$

so that the vectors are linearly dependent by our first definition.

How can we determine if the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly dependent or linearly independent? Here is one way, essentially that discussed in Boyce and DiPrima. Suppose that these are column vectors with n components, and build a matrix A with these vectors

as columns: $A = (\mathbf{x}_1 \, \mathbf{x}_2 \, \ldots \, \mathbf{x}_k)$. The matrix A is $n \times k$. To say that (11) holds is just to say that $A\mathbf{c} = \mathbf{0}$, where $\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$. This means that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly dependent—i.e., that (11) holds with the c_i not all zero—if the system of equations $A\mathbf{c} = \mathbf{0}$ has a nontrivial solution for $\mathbf{c}$. One can determine whether or not it does by reducing A to REF or RREF.

Finally, suppose we have n vectors, each with n components, and want to know if they are linearly independent (this is the case considered by Boyce and DiPrima). Then the matrix A is an $n \times n$ square matrix, and we can study it via the ideas of the previous section. The system $A\mathbf{c} = \mathbf{0}$ has no nontrivial solution if and only if we are in Case 1 (this is condition **C3** for being in case 1), i.e., if the matrix A satisfies any of the conditions **C1–C7**. Note that this means that we could add another condition to the list **C1–C7**, equivalent to all the rest:

C8. The columns of A are linearly independent vectors.

We summarize:

Principle 5: Linear independence. The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly dependent if the system of equations $A\mathbf{c} = \mathbf{0}$, where $A = (\mathbf{x}_1 \, \mathbf{x}_2 \, \ldots \, \mathbf{x}_k)$, has a nontrivial solution. The vectors are linearly independent if the system $A\mathbf{c} = \mathbf{0}$ has only the trivial solution $\mathbf{c} = \mathbf{0}$. If $k = n$ and the vectors are column vectors with n components, then they are linearly independent if and only if the matrix A satisfies any of the conditions **C1–C7** of Section 4.

Remark 1: In Principle 3 on page 8 we found that $k = n - \mathrm{rank}(A) = n - r$ vectors are needed to express every solution of the equations $A\mathbf{x} = \mathbf{b}$, and observed that row reduction produced the needed vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$. We want to observe here that these k vectors are **linearly independent**. To see this, consider Example 4 and form a linear combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ produced there:

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ -2c_2 - c_3 \\ c_2 \\ -3c_3 \\ c_3 \end{pmatrix}.$$

By looking at the first, third, and fifth components of the final form of this vector we see that if $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = 0$ then necessarily $c_1 = c_2 = c_3 = 0$, and this is precisely linear independence of $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$. The pattern is the same for any system $A\mathbf{x} = \mathbf{b}$.

6. Linear independence of functions

In the previous section we discussed what it means to say that a collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is linearly independent. We will also use a slightly different concept, that of linear independence of *functions*. Consider a collection of functions defined on some interval I; these could either be scalar-valued functions $f_1(t), \ldots, f_k(t)$, say

$$f_1(t) = 1, \quad f_2(t) = \sin^2 t, \quad f_3(t) = \cos^2 t, \tag{12}$$

or could be vector-valued functions $\mathbf{x}^{(1)}(t)$, ..., $\mathbf{x}^{(k)}(t)$, all with the same number of components, say

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}. \tag{13}$$

(We will usually use the notation of vector-valued functions in general discussions, to avoid separate consideration of several cases. One can of course think of scalar functions as vector functions having only one component.) We say that the collection is *linearly independent on I* if no one of these functions may be expressed as a linear combination, with constant coefficients, of the others. Alternatively, the collection is linearly independent if whenever a linear combination of the functions, with constant coefficients, is zero for all t in I, then necessarily all the coefficients are zero:

$$c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0} \quad \text{for all } t \text{ in } I \quad \Rightarrow \quad c_1 = \cdots = c_k = 0.$$

Equivalence of these two definitions of linear independence is proved just as was the corresponding equivalence for vectors, above. A set of functions which is not linearly independent is *linearly dependent*.

Example 6: (a) The three (scalar) functions $f_1(t) = 1$, $f_2(t) = \sin^2 t$, and $f_3(t) = \cos^2 t$ of (12) are linearly dependent on any interval I, since we always have $1 - \sin^2 t - \cos^2 t = 0$, that is, $c_1f_1(t) + c_2f_2(t) + c_3f_3(t) = 0$ for $c_1 = 1$, $c_2 = c_3 = -1$.

(b) On the other hand, the three functions $g_1(t) = 1$, $g_2(t) = \sin t$, and $g_3(t) = \cos t$ are linearly dependent on any interval I. Here is one way to see this, assuming that the interval contains at least the points 0, $\pi/2$, and π. Suppose that $c_1g_1(t) + c_2g_2(t) + c_3g_3(t) = 0$ for all t; then plugging in successively these three values of t gives

$$c_1 + c_3 = 0, \quad (t = 0); \qquad c_1 + c_2 = 0, \quad (t = \pi/2); \qquad c_1 - c_3 = 0, \quad (t = \pi),$$

and the only solution of these equations is $c_1 = c_2 = c_3 = 0$.

(c) The two vector functions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ of (13) are linearly independent on any interval I. For if $c_1\mathbf{x}^{(1)}(t) + c_2x^{(2)}(t) = 0$ for all t then one may plug in any two distinct values of t in the interval, say $t = r$ and $t = s$, to find

$$c_1\mathbf{x}^{(1)}(r) + c_2\mathbf{x}^{(2)}(r) = c_1\begin{pmatrix} 1 \\ r \end{pmatrix} + c_2\begin{pmatrix} e^r \\ re^r \end{pmatrix} = \begin{pmatrix} c_1 + c_2e^r \\ r(c_1 + c_2e^r) \end{pmatrix} = 0$$

$$c_1\mathbf{x}^{(1)}(s) + c_2\mathbf{x}^{(2)}(s) = c_1\begin{pmatrix} 1 \\ s \end{pmatrix} + c_2\begin{pmatrix} e^s \\ se^s \end{pmatrix} = \begin{pmatrix} c_1 + c_2e^s \\ s(c_1 + c_2e^s) \end{pmatrix} = 0$$

In particular, $c_1 + c_2e^r = 0$ and $c_1 + c_2e^s = 0$, and since $r \neq s$ these equations imply $c_1 = c_2 = 0$.

(d) Consider again the vector functions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ of (13). From (c) we know that these are independent as functions on and interval. However, for *fixed* t, say $t = t_0$, the two vectors $\mathbf{x}^{(1)}(t_0) = \begin{pmatrix} 1 \\ t_0 \end{pmatrix}$ and $\mathbf{x}^{(2)}(t_0) = e^{t_0}\begin{pmatrix} 1 \\ t_0 \end{pmatrix}$ are linearly *dependent*, since they

are proportional (with proportionality constant e^{t_0}). This is Exercise 7.3.15 of Boyce and DiPrima.

Remark 2: We have a systematic method of determining whether or not a set of vectors is linearly independent, since (see Principle 5) this reduces to solving a system of linear equations. There is no corresponding simple method for checking linear dependence or independence of functions. However, the method used in Example 6(b,c), of plugging in a few points, is very useful to show linear independence. To show linear dependence one can, for examples considered in these notes, usually see by inspection how to express one of the functions in as a linear combination of some of the others.

7. Eigenvalues and eigenvectors

The discussion of eigenvalues and eigenvectors in Boyce and DiPrima Section 7.3 (pages 379–383) is fine for our purposes.

8. Linear systems of ordinary differential equations

In this section we discuss the solutions of first order linear systems of ordinary differential equations. This is not a thorough presentation of the entire subject; rather, we want to summarize the essential **general** properties of the problem, and in particular those aspects related to the **linearity** of the equations. In doing so we will emphasize the parallels with the solution of algebraic linear equations, discussed earlier in these notes.

The system we will study is

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{f}, \tag{14}$$

where $\mathbf{x}$ and $\mathbf{f}$ are vector functions of t, with $\mathbf{f}$ known and $\mathbf{x}$ unknown—$\mathbf{x}$ is the *dependent variable*—and $\mathbf{P}$ is a known matrix function of t:

$$\mathbf{x} = \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{f} = \mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \quad \mathbf{P} = \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix}.$$

Note that P is a **square** matrix. Throughout we assume that all the functions $p_{ij}(t)$ and $f_i(t)$ are continuous functions of t for t in some interval I, and we are look for a solution or solutions $\mathbf{x}(t)$ defined on this interval.

The homogeneous problem. Let us first consider the homogeneous case in which $\mathbf{f} = 0$, so that our system becomes

$$\mathbf{x}' = \mathbf{P}\mathbf{x}. \tag{15}$$

Just as for a homogeneous system of (algebraic) linear equations (see Principle 1 on page 1), we have

> **Principle 6: Superposition for linear ODEs.** If $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, ..., $\mathbf{x}^{(k)}(t)$ are solutions of (15), and c_1, c_2, ..., c_k are constants, then
>
> $$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \cdots + c_k \mathbf{x}^{(k)}(t) \tag{16}$$
>
> is also a solution of (15).

The principle of superposition tells us how to build new solutions from solutions we already have; $\mathbf{x}(t)$ in (16) is called a *superposition* or *linear combination* of the solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$. The principle is of fundamental importance; let us verify it. Since we want to check a claim that $\mathbf{x}$ is a solution of (15), we just plug $\mathbf{x}$ into (15) and see if the equation is satisfied. We need to use the linearity of differentiation and of matrix multiplication:

$$\mathbf{x}' = (c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \cdots + c_k \mathbf{x}^{(k)})' = c_1 \mathbf{x}^{(1)\prime} + c_2 \mathbf{x}^{(2)\prime} + \cdots + c_k \mathbf{x}^{(k)\prime}, \tag{17}$$

$$\mathbf{P}\mathbf{x} = \mathbf{P}(c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \cdots + c_k \mathbf{x}^{(k)}) = c_1 \mathbf{P}\mathbf{x}^{(1)} + c_2 \mathbf{P}\mathbf{x}^{(2)} + \cdots + c_k \mathbf{P}\mathbf{x}^{(k)}. \tag{18}$$

Then since $\mathbf{x}^{(i)\prime} - \mathbf{P}\mathbf{x}^{(i)} = \mathbf{0}$ for each i,

$$\begin{aligned}
\mathbf{x}' - \mathbf{P}\mathbf{x} &= (c_1 \mathbf{x}^{(1)\prime} + c_2 \mathbf{x}^{(2)\prime} + \cdots + c_k \mathbf{x}^{(k)\prime}) - (c_1 \mathbf{P}\mathbf{x}^{(1)} + c_2 \mathbf{P}\mathbf{x}^{(2)} + \cdots + c_k \mathbf{P}\mathbf{x}^{(k)}) \\
&= c_1 (\mathbf{x}^{(1)\prime} - \mathbf{P}\mathbf{x}^{(1)}) + c_2 (\mathbf{x}^{(2)\prime} - \mathbf{P}\mathbf{x}^{(2)}) + \cdots + c_k (\mathbf{x}^{(k)\prime} - \mathbf{P}\mathbf{x}^{(k)}) = \mathbf{0},
\end{aligned} \tag{19}$$

that is, $\mathbf{x}$ satisfies (15).

The inhomogeneous problem. We now consider general case of (14), in which $\mathbf{f}$ may or may not be zero. Again the result is parallel to the result for algebraic systems, given in Principle 2 on page 2.

> **Principle 7: Inhomogeneous linear ODEs.** Suppose that $\mathbf{X}(t)$ is some (particular) solution of the inhomogeneous equation (14). Then every solution of this equation is of the form
>
> $$\mathbf{x}(t) = \mathbf{X}(t) + \mathbf{x}_h(t), \tag{20}$$
>
> where $\mathbf{x}_h(t)$ is a solution of the homogeneous system (15).

To verify this principle, one checks two things: first, that (20) is a solution of (14), and second, that if $\hat{\mathbf{x}}$ is any solution of (14), then $\hat{\mathbf{x}} - \mathbf{X}$ satisfies the homogeneous equation (15). Both of these are checked by substituting the purported solutions into the relevant equations.

Linear independence and the general solution. The superposition principle tells us how to build many solutions of the homogeneous equation (and thus, through (20), of the inhomogeneous equation), once we have some "building blocks": the solutions $x^{(i)}(t)$ used in (16). Now we ask: is it possible to get all solutions this way, and if so, how many building blocks do we need? For the algebraic systems this question is answered by Principle 3 on page 8 and the Remark 1 on page 11: we need $k = n - r$ linearly independent

solutions as building blocks, where n is the total number of unknowns and r is the rank of the coefficient matrix. The next principle gives us the corresponding information for the homogeneous and inhomogeneous ODE systems (15) and (14).

Principle 8: The general solution of ODEs. Suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, ..., $\mathbf{x}^{(n)}$ are n solutions of (15), linearly independent on the interval I. Then every solution $\mathbf{x}$ of (15) is of the form

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \tag{21}$$

for some constants c_1, c_2, ..., c_n. Moreover, if $\mathbf{X}(t)$ is some (particular) solution of the inhomogeneous equation (14), then every solution of (14) is of the form

$$\mathbf{x}(t) = \mathbf{X}(t) + c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t), \tag{22}$$

where $\mathbf{x}_h(t)$ is a solution of the homogeneous system (15).

It is important to realize that there are three conditions here on the vector functions $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ which are necessary to guarantee that the general solution has the form (21). First, the $\mathbf{x}^{(i)}$ must themselves be solutions of (15). Second, there must be exactly n solutions: *the same number of solutions as the number of components in the vectors, i.e., as the number of different first order equations represented in vector form in (15).* Third, these n solutions must be *linearly independent.*

To apply Principle 8 it is of course necessary to determine whether some set $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ of solutions of (15) is linearly independent. As indicated in Remark 2 on page 13 there is in general no simple method to determine whether a set of function is or is not linearly independent. However, when the functions in questions are all solutions of (15) there is such a method; one uses the *Wronskian* of these solutions, that is, the determinant whose columns are the solutions:

$$W(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)})(t) = W(t) = \begin{vmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \cdots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \cdots & x_2^{(n)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \cdots & x_n^{(n)}(t) \end{vmatrix}.$$

A nonzero Wronskian corresponds to linear independence of the solutions; for more deail, see Principle 9 on page 16. In paricular, if we find n solutions of the homogeneous problem (15) whose Wronskian does not vanish (and it suffices to check the Wronskian at one point) then these can be used to build the general solution, as in (21).

Principle 9: The Wronskian. Suppose that we have precisely n functions $\mathbf{x}^{(1)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$, all of which are solutions of the homogeneous equation (15). Then either

(i) $W(t) = 0$ for all t in I, in which case the vector functions $\mathbf{x}^{(1)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ are linearly dependent on I and, for any fixed point t_0 in I, the (constant) vectors $\mathbf{x}^{(1)}(t_0)$, $\ldots$, $\mathbf{x}^{(n)}(t_0)$ are linearly dependent, or

(ii) $W(t) \neq 0$ for all t in I, in which case the vector functions $\mathbf{x}^{(1)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ are linearly independent on I and, for any fixed point t_0 in I, the (constant) vectors $\mathbf{x}^{(1)}(t_0)$, $\ldots$, $\mathbf{x}^{(n)}(t_0)$ are linearly independent.

9. Exercises

1. Boyce and DiPrima, Section 7.3, problems 1, 2, and 3 (already assigned). Do these problems specifically by the methods used in these notes, that is, by introducing the augmented matrix and then reducing it to row-echelon form or reduced row-echelon form.

2. In (a)–(d) below we suppose that we have been given a system of equations $A\mathbf{x} = \mathbf{b}$ and that we have already reduced the augmented matrix $(A\,|\,\mathbf{b})$ to the row-echelon form $(R\,|\,\mathbf{e})$ given. In each case, determine whether or not the original equations have a solution. If they do have a solution, determine whether or not it is unique and, if it is not unique, how many free parameters there are; then write the solution explicitly in the form (9).

(a) $\quad (R\,|\,\mathbf{e}) = \begin{pmatrix} 1 & 5 & -3 & 2 & 8 & | & 2 \\ 0 & 0 & 1 & -1 & 0 & | & 3 \\ 0 & 0 & 0 & 0 & 0 & | & 1 \end{pmatrix}$

(b) $\quad (R\,|\,\mathbf{e}) = \begin{pmatrix} 1 & 0 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 1 & | & 4 \end{pmatrix}$

(c) $\quad (R\,|\,\mathbf{e}) = \begin{pmatrix} 1 & -1 & 2 & 3 & | & 2 \\ 0 & 1 & 1 & -1 & | & -1 \\ 0 & 0 & 1 & -4 & | & 3 \\ 0 & 0 & 0 & 1 & | & 4 \end{pmatrix}$

(d) $\quad (R\,|\,\mathbf{e}) = \begin{pmatrix} 0 & 1 & 2 & 0 & -2 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 & | & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$

3. In each part below, give a $m \times n$ matrix R *in reduced row-echelon form* satisfying the given condition, or explain briefly why it is impossible to do so.

(a) $m = 3$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{e}$ has a solution for all $\mathbf{e}$.

(b) $m = 3$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has a unique solution.

(c) $m = 4$, $n = 3$, and the equation $R\mathbf{x} = \mathbf{e}$ has a solution for all $\mathbf{e}$.

(d) $m = 4$, $n = 3$, and the equation $R\mathbf{x} = \mathbf{0}$ has a unique solution.

(e) $m = 4$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has no solution.

(f) $m = 4$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

(g) $m = 4$, $n = 4$, and for every $\mathbf{e}$ the equations $R\mathbf{x} = \mathbf{e}$ have a solution containing a free parameter.

4. Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of $A\mathbf{x} = 0$ and that $\mathbf{X}$ is a solution of $A\mathbf{x} = \mathbf{b}$. *Without looking at the these notes or the book*, show that for any constants c_1 and c_2, $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is a solution of $A\mathbf{x} = 0$ and that $\mathbf{X} + c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is a solution of $A\mathbf{x} = \mathbf{b}$.

5. In each part below, show that the given functions are linearly independent or linearly dependent, as indicated. You may take the interval in question to be $(-\infty, \infty)$. See Remark 2 on page 13.

(a) $f_1(t) = 2$, $f_2(t) = 3t$, $f_3(t) = 1 - 7t$ (linearly dependent).

(b) $g_1(t) = t$, $g_2(t) = t^2$ (linearly independent).

(c) $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{x}^{(2)}(t) = \begin{pmatrix} t \\ t \end{pmatrix}$ (linearly independent). Show also that the vectors are linearly dependent (as vectors) for $t = 0$ and $t = 1$ but not for $t = 2$.

Hints: 2. (a) no solution, (b),(c) unique solution, (d) solution with 3 parameters.   4.(b),(c),(e),(g) impossible.