OPTIONAL READING SEMINAR ON FOURIER ANALYSIS

The plan is to review certain sections of the textbook by Stein and Shakarchi, and have some of you present some theorems from the book. This will hopefully help all of us understand the material better, and stimulate discussion.

Each talk is 15 minutes long. Here are the outlines of each of the talks. They will be updated as we progress.

Talk 1: Properties of good kernels, Sophie Wang

(1) What a family of good kernels is.
(2) An example: $K_n(x) = n$ if $|x| < 1/n$, 0 otherwise, on $[0, 1]$; this is one of the simplest examples of a good kernel.
(3) The first part of Theorem 4.1 in Chapter 2 and its proof, namely that if f is continuous at x, then $f * K_n(x) \to f(x)$ as $n \to \infty$. One should try to break the proof into manageable steps and explain why each step should be done in the way given in the book.
(4) What the above proof is doing when $K_n(x)$ is as in the example above, and observe how the proof can be (slightly) simplified in that case (because by being more concrete this way hopefully we will get to know the proof better).

You may find the following principles useful in explaining the above proofs:

(a) In estimating an integral, if you are integrating a small quantity, then the integral is small;
(b) If you are integrating over a small interval, then the integral is small as long as the integrand is not too big.

Talk 2: Partial differential equations, Thomas Goller

An overview of some of the partial differential equations that we have come across in this course, namely

(1) Heat equation on the unit circle
(2) Laplace equation on the unit disk
(3) Heat equation on the real line
(4) Laplace equation on the upper half plane

Write down what the equations are (including the appropriate boundary conditions), and discuss the kernels that one use to solve these equations. It may be useful to collect these things in a table form.

Talk 3: The heat equation, Paul Gustafson

(1) The derivation of the heat kernel on the real line on p.146 of the book using Fourier transform and by solving an ODE
(2) The proof of the first part of Theorem 2.1 in Chapter 5, namely how one can differentiate the integral to show that

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-4\pi^2|\xi|^2 t} e^{2\pi i x\xi} d\xi$$

actually solves the heat equation on $\mathbb{R}$. It should be useful here to understand and apply the proof of Proposition 1.2(v) of Chapter 5.

Talk 4: Harmonic functions, Valeri Karpov

Prove that if u is a C^2 harmonic function on the whole $\mathbb{R}^2$ and $|u(x)| \to 0$ as $|x| \to \infty$, then u is identically zero.

This is a simpler statement than Theorem 2.7 of chapter 5, and it follows easily from the mean-value property of harmonic functions (Lemma 2.8). This should be pretty quick, and after this we may either discuss the proof of Theorem 2.7 itself, or the proof of Lemma 2.8.

Talk 5: An application of Poisson summation: Dan Li

Prove that the heat kernel on the unit circle is a good kernel using the Poisson summation formula (Theorem 3.3 and Corollary 3.4 of Chatper 5), and supplement that by a short discussion of why the Poisson summation formula is valid in that case.

Talk 6: Finite speed of propagation: Will Pickering

Discuss the concepts of light cones and finite speeds of propagation. These concepts are best understood by looking at the solution formula of the wave equation that involves spherical averages. If time permits, could mention also that the solution formula using Fourier transform is good for proving conservation of energy.

Talk 7: Inversion of Radon transform: Bill Harvey

Prove the inversion formula for the Radon transform in $\mathbb{R}^3$. Discuss the difficulty for inverting the Radon transform in $\mathbb{R}^2$.

Talk 8: Peter-Weyl theorem for finite abelian groups: Kevin Schenthal

Prove that any function on a finite abelian group G is a linear combination of the characters of G. This involves interpreting characters of G as eigenfunctions of the translation operators $\tau_g(f)(x) = f(xg^{-1})$, and uses the fact that $\{\tau_g \colon g \in G\}$ is a family of commuting unitary operators on the space of all functions on G.

Talk 9: Bessel functions: Ram Shankar

Discuss the equivalent definitions of the Bessel functions. Also discuss their relevance in the study of the Fourier transform of radial functions on $\mathbb{R}^d$.