THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH4210 Financial Mathematics

Homework 4

Due Date: April 27, 2018

Name: ________________________ Student ID.: ________________________

I declare that the assignment here submitted is original except for source material explicitly acknowledged. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the website http://www.cuhk.edu.cn/departsite/ar/en/Academic.html/

________________________ ________________________
Signature Date

For lecturer/TA's use only

1		4		7	
2		5			
3		6			

Total

General Regulations

- Assignments should be printed and hardcopies should be submitted on the due date to the lecturer by end of the lecture. Assignments will not be accepted by e-mail.
- Late assignments will receive a grade of 0.
- Print out the cover sheet (i.e. the first page of this document), and sign and date the statement of Academic Honesty.
- All the pages of your assignment MUST BE STAPLED together (NOT paper-clipped), with the cover sheet as the first page. Failure to comply with these instructions will result in a 10-point deduction).
- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read).
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.
- Your graphical solutions, dots and lines, must be readable and neat. Dot and line scale gradations should be uniform, labeled with whole numbers and text, and be readable (avoid over-crowding). Label your graphs.

Instructions for Homework 3

Please attempt to solve all the problems.

- Your solutions of problems 1 - 7 are to be submitted.
- We strongly recommend that you study problems 8 - 12, though you are not required to submit their solutions. But if you do, we will correct them for you.
- Please try to use **R** to get/read the data from the csv file and do some calculations.

[**Objective:**] The types of questions on this homework assignment are classified into three groups:

- Theoretical part;
- Computational part;
- Usage of Computing Software, namely `MATLAB` and/or **R**.

The objective of this homework assignment is to learn how to understand:

- Ito's formula and stochastic differential equations;
- the Binomial model for option pricing;
- the applications of the Black-Scholes formula;
- Taylor's formula and Taylor Series for the approximation of the Black-Scholes formula;
- Finite difference for the Black-Scholes formula;
- the concept of risk-neutral pricing;
- the concept of implied volatility;
- the concepts of Greeks and hedging; and
- the applications of Cox-Ross-Rubinstein parameterization.

1.

Problem Description:

In what follows the stock price $S(t)$ obeys the stochastic differential equation

$$dS = \mu S dt + \sigma S dW,$$

where μ is the expected return, σ is the volatility, and $W(t)$ is a standard Wiener process (Brownian motion).

Question:

(a) Suppose that the expected return from a stock is 14% per annum and the volatility is 20%. The initial stock price is \$90. By using $dW \approx X\sqrt{\Delta t}$, where X is $\mathcal{N}(0,1)$, calculate the increase ΔS in the stock price during three days.

(b) Suppose that a stock price has an expected return of 36% per annum and a volatility of 40%. When the stock price at the end of a certain day is \$80, calculate the following

  i. the expected stock price at the end of the next day; and

  ii. the standard deviation of the stock price at the end of the next day.

(c) If $dS = \mu S dt + \sigma S dW$, and A and n are constants, find the SDE satisfied by (use Ito's Lemma):

  i. (Optional) $f(S) = AS + t^2$;

  ii. (Optional) $f(S) = S^{1/2}$;

  iii. $f(S) = S^n$.

2. (a) Two-step binomial tree Stock price starts at \$20 and in each of the next two three-month periods may go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding. What is the value of a six-month European call option with a strike price of \$21?

   (b) Risk-Neutral World Consider a two-step binomial tree with $u = 1.2$ and $d = 0.9$. Initial stock price $S_0 = S(0) = 40$. The interest rate is 5% per annum, compounded continuously. Work out the probability distribution of the stock price in six months in a risk-neutral world.

   (c) Binomial tree matching volatility σ with u and d. By using three equations, we have:

$$qu + (1-q)d = e^{\mu \Delta t};$$

$$qu^2 + (1-q)d^2 - (qu + (1-q)d)^2 = \sigma^2 \Delta t,$$

   and

$$d = \frac{1}{u}.$$

   Find that

$$u = e^{\sigma\sqrt{\Delta t}} \approx 1 + \sigma\sqrt{\Delta t}.$$

3. We take a short position in a European Call option with maturity 4 months and with strike of 20 dollars, having a stock with current price of 20 dollars as the underlying. In the next 4 months, the stock price may increase by a growth factor $u = 1.2$ or decrease by a factor $d = 0.8$. The risk-free interest rate available on the market is 4% per year.

   (a) Find the replicating strategy of the option.

   (b) Compute the initial price of the option.

   (c) Suppose now that the stock price does not follow a binomial model any more, but that in 4 months it may either increase by a growth factor $u = 1.2$, decrease by a factor $d = 0.8$ or remain unchanged. Discuss whether it is still possible to replicate the Call option above with only the underlying and with cash (to be deposited or borrowed).

4. In the Cox-Ross-Rubinstein parameterization for a binomial tree, the up and down factors u and d, and the risk-neutral probability of the price going up during one time-step are:

$$u = A + \sqrt{A^2 - 1}; \tag{1}$$

$$d = A - \sqrt{A^2 - 1}; \tag{2}$$

$$p = \frac{e^{r\delta t} - d}{u - d}, \tag{3}$$

where

$$A = \frac{1}{2}\left(e^{-r\delta t} + e^{(r+\sigma^2)\times\delta t}\right).$$

Use Taylor expansions to show that, for a small time step $\delta t, u, d$ and p may be approximated by

$$u = e^{\sigma\sqrt{\delta t}}; \tag{4}$$

$$d = e^{-\sigma\sqrt{\delta t}}; \tag{5}$$

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\delta t}. \tag{6}$$

In other words, write the Taylor expansions for (1) - (3) and for (4) - (6) and show they are identical if all the terms of order $\mathcal{O}(\delta t)$ and smaller are neglected.

5. Consider a 3-period Cox-Ross-Rubinstein model. The annual interest rate is $r = 0.05$ (discrete), $u = 1.1$ and $d = 0.9$. The initial price of the stock is $S(0) = 100$. The time horizon is $T = 3$ years.

   (a) Calculate the risk-neutral probability and the stock prices at each node in the binomial tree (correct up to 2 decimal places after the decimal point).

   (b) Calculate the value of the European option with payoff:

$$P(T) = \begin{cases} \sup_{0\le t\le T} S_t - S_T, & \text{if } S_t < 100 \ \forall t; \\ 0, & \text{otherwise} \end{cases}$$

   (c) Find a replicating portfolio for the above option for the first trading period.

6. Assume a 3-period Cox-Ross-Rubinstein mode. The annual interest rate with continuous compounding is $r = 0.06$. The volatility of the stock is $\sigma = 0.2$ with a price of $S(0) = 100$. Furthermore, there exists an American Put with maturity date $T = 1$ and strike $K = 90$.

   (a) Calculate the risk-neutral probability and the stock prices at each node in the binomial tree (correct up to 2 decimal places after the decimal point).

   (b) Calculate the value of the American Put for all nodes in the tree.

   (c) What is the optimal stopping time? Justify your answer.

7. Calculate the price of a three-month European call option on a stock with a strike price \$60 when the current stock price is \$80. The risk-free interest rate is 10% per annum. The volatility is 30%. Use a scientific calculator and the cumulative distribution table to get the solution to $\mathcal{N}(\cdot)$ and compare these two results. Are they the same? If not, what is the magnitude of their difference?

8. (Optional) Recall that finding the implied volatility from the given price of a call option is equivalent to solving the nonlinear problem $f(x) = 0$, where

$$f(x) = Se^{-qT}N(d_1(x)) - Ke^{-rT}N(d_2(x)) - C.$$

Answer the following questions:

(a) Show that

$$\lim_{x \to +\infty} d_1(x) = +\infty$$

and

$$\lim_{x \to +\infty} d_2(x) = -\infty$$

and conclude that

$$\lim_{x \to +\infty} f(x) = Se^{-qT} - C.$$

(b) Show that

$$\lim_{x \to 0} d_1(x) = \lim_{x \to 0} d_2(x) = \begin{cases} -\infty, & \text{if } Se^{(r-q)T} < K; \\ 0, & \text{if } Se^{(r-q)T} = K; \\ +\infty, & \text{if } Se^{(r-q)T} > K. \end{cases}$$

Here $F = Se^{(r-q)T}$ is the forward price.

Conclude that

$$\lim_{x \to 0} f(x) = \begin{cases} -C, & \text{if } Se^{(r-q)T} \le K; \\ Se^{-qT} - Ke^{-rT} - C, & \text{if } Se^{(r-q)T} > K. \end{cases}$$

(c) (Optional) Show that $f(x)$ is a strictly increasing function and

$$\begin{cases} -C < f(x) < Se^{-qT} - C & \text{if } Se^{(r-q)T} \le K; \\ Se^{-qT} - Ke^{-rT} - C < f(x) < Se^{-qT} - C, & \text{if } Se^{(r-q)T} > K. \end{cases}$$

(d) For what range of call option values does the problem $f(x) = 0$ have a positive solution? Compare your result to the range

$$Se^{-qT} - Ke^{-rT} < C < Se^{-qT}$$

required for obtaining a positive implied volatility for a value C of the call option.

9. (Optional) Consider the Black-Scholes formula for the price of a European call option as a function of a single variable S (i.e., treat all the other inputs as constant). Write down a second order Taylor polynomial around $S(0)$ for $C(S)$ in terms of Δ and Γ.

10. (Optional) Consider a function $f(x)$.

    (a) Write down a finite difference scheme to approximate $f^{(3)}(x)$. What order is this scheme?

    (b) We would like to write down a fourth order finite difference scheme for $f'(x)$ using the points $f(x-2\Delta x)$, $f(x-\Delta x)$, $f(x)$, $f(x+\Delta x)$, and $f(x+2\Delta x)$. Use a Taylor series to construct a linear system for the coefficients of each term. Solve this system in whatever way you please to find the scheme.

    (c) Implement the scheme from (b) in **R**. Use it to compute the Delta of the Black-Scholes formula at $S = 50$ for the parameters used in the lecture. Show that the convergence is $\mathcal{O}(\Delta S^4)$. Changing the step size by a factor of 10 reduces the error by a factor of 10^4.

11. (Optional) Select another of the Greeks.

    (a) Implement first, second, and fourth order finite difference schemes in **R**, and use these to approximate your chosen Greek along a range of values you deem appropriate. Plot these approximations on the same graph alongside the exact answer.

    (b) Show numerically that your schemes converge at the correct order.

    (c) Repeat (a) and (b) for all the other Greeks!

12. (Optional)

(a) Implement the right endpoint rule as a function in **R**. Use it to compute

$$\int_{-1}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

for several discretizations h. Compare your results to the exact answer and verify the order of convergence of the right endpoint method. Scales linearly therefore it is order $\mathcal{O}(h)$.

(b) How might you use a numerical integrator to compute the integral

$$\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx?$$

Implement a function that takes as input a and the number of subintervals n and numerically approximates the integral using a scheme of your choice.

(c) The Black-Scholes formula for a European call option is:

$$C(\cdot) = Se^{-q(T-t)}\mathcal{N}(d_+) - Ke^{-r(T-t)}\mathcal{N}(d_-)$$

where

$$\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx,$$

$$d_+ = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_- = d_+ - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Using your results from (b), write a function in **R** that takes as input S, T, t, K, r, σ, and q and outputs the Black-Scholes call option price with all integrals computed numerically.

(d) Use one of your finite difference schemes to approximate one of the Greeks using the function from (c) to compute the data points. Compare this to the exact answer, and comment on how the numerical integration affected your accuracy.