THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH4210 Financial Mathematics

Homework 3

Due Date: April 6, 2018

Name: ________________________ Student ID.: ________________________

I declare that the assignment here submitted is original except for source material explicitly acknowledged. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the website http://www.cuhk.edu.cn/departsite/ar/en/Academic.html/

______________________________ ______________________________
Signature Date

For lecturer/TA's use only

1		4a		5a	
2		4b		5b	
3					

Total

General Regulations

- Assignments should be printed and hardcopies should be submitted on the due date to the lecturer by end of the lecture. Assignments will not be accepted by e-mail.
- Late assignments will receive a grade of 0.
- Print out the cover sheet (i.e. the first page of this document), and sign and date the statement of Academic Honesty.
- All the pages of your assignment MUST BE STAPLED together (NOT paper-clipped), with the cover sheet as the first page. Failure to comply with these instructions will result in a 10-point deduction).
- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read).
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.
- Your graphical solutions, dots and lines, must be readable and neat. Dot and line scale gradations should be uniform, labeled with whole numbers and text, and be readable (avoid over-crowding). Label your graphs.

Instructions for Homework 3

Please attempt to solve all the problems.

- Your solutions of problems 1 - 5 are to be submitted.
- We strongly recommend that you study problems 6 - 7, though you are not required to submit their solutions. But if you do, we will correct them for you.
- Please try to use **R** to get/read the data from the csv file and do some calculations.

[**Objective:**] The types of questions on this homework assignment are classified into three groups:

- Theoretical;
- Computational;
- Usage of Computing Software, namely `MATLAB` and/or **R**.

The objective of this homework assignment is to learn how to understand:

- the usage of the put-call parity;
- the properties of the Black-Scholes formula;
- the applications of the Black-Scholes formula;
- the sensitivity studies of the Black-Scholes formula;
- the concepts of lognormal distributions;
- the estimation of the historical volatility using the statistical method;
- the estimation of the implied volatilities using the least-squares method;
- the concept of risk-neutral probability.

1. Consider the following conditions:
   - The Black-Scholes framework holds.
   - The stock is currently selling for 20.
   - The stock's volatility is 24%.
   - The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.
   - The continuously compounded risk-free interest rate is 5%.

   You are considering the purchase of 100 units of a 3−month 25-strike European call option on a stock. What is the price of the block of 100 options?

2. Compute the Greeks indicated the following table for a European call option with the strike $K = \$24$ and expiration $T = 0.3$ on a non-dividend paying stock with current price $S = \$25$ and volatility $\sigma = 0.25$. The risk-free rate is $r = 0.05$.

Quantity	Call
Black Scholes price	
delta $\Delta = \dfrac{\partial C}{\partial S}$	
gamma $\Gamma = \dfrac{\partial^2 C}{\partial S^2}$	
theta $\Theta = \dfrac{\partial C}{\partial t}$	
rho $\rho = \dfrac{\partial C}{\partial r}$	
vega $\dfrac{\partial C}{\partial \sigma}$	

Please show your calculations clearly.

3. Properties of Black-Scholes Formula

Problem Description: Consider a stock paying no income. Assume the Black-Scholes model of the stock. For a non-dividend-paying asset, the Black-Scholes formula for a European call is

$$C^E(S,t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Recall the bounds on a European call we obtained in **Lecture Note 7, Proposition 1**, which states:

The prices of a European call on a stock paying no dividends satisfy the inequality:

$$\max(S(t) - Ke^{-r(T-t)}, 0) \le C^E(S,t) \le S(t).$$

These bounds are illustrated when we plot the price of a 1-year 80-strike call under the Black-Scholes model for various values of the volatility σ.

Question: Show that

$$C^E(S,t) \to \max\{S(t) - Ke^{-r(T-t)}, 0\} \quad \text{as} \quad \sigma \to 0.$$

Hint:

$$\lim_{\sigma\to 0} d_1 = \begin{cases} \_\_\_\_\_\_ & \text{if } \ln\left(\frac{S(t)}{K}\right) + r(T-t) > 0; \\ \_\_\_\_\_\_ & \text{if } \ln\left(\frac{S(t)}{K}\right) + r(T-t) = 0; \\ \_\_\_\_\_\_ & \text{if } \ln\left(\frac{S(t)}{K}\right) + r(T-t) < 0; \end{cases}$$

and

$$\lim_{\sigma\to 0} N(d_1) = \lim_{\sigma\to 0} P(X \le d_1) = \begin{cases} \_\_\_\_\_\_ & \text{if } S_t > Ke^{-r(T-t)}; \\ \_\_\_\_\_\_ & \text{if } S_t = Ke^{-r(T-t)}; \\ \_\_\_\_\_\_ & \text{if } S_t < Ke^{-r(T-t)}, \end{cases}$$

where $X \sim N(0,1)$.

4. <u>**Volatility**</u>

Preparation: Assuming the Black-Scholes model for the stock, we have, for any times $t < T$,

$$\ln\left(\frac{S(T)}{S(t)}\right) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$$

Let $S(i\Delta t)$ be the stock price on day i, where $\Delta t = \frac{1}{365}$ years = 1 day.

Define

$$X_i = \ln\left(\frac{S(i\Delta t)}{S((i-1)\Delta t)}\right).$$

Then

$$X_i = \ln\left(\frac{S(i\Delta t)}{S((i-1)\Delta t)}\right)$$
$$= \ln(S(i\Delta t)) - \ln(S((i-1)\Delta t)) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$$

Therefore

$$\sigma^2 \Delta T = \text{Var}(X_i).$$

Question: Do the following:

(a) Use the stock price data given in the Excel file, called `Question_1.xlsx`, to estimate the historical volatility σ of the stock.

<u>**Procedure to Estimate σ From Data**</u>:

- Compute
$$X_i = \ln\left(\frac{S(i\Delta t)}{S((i-1)\Delta t)}\right)$$
for day $i = 1, 2, \cdots, 35$.
- Compute the sample variance of the data points: $X_1, X_2, \cdots, X_n$:
$$s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$
where
$$\bar{X} = \frac{1}{n-1}\sum_{i=1}^{n} X_i$$
Then
$$s_n^2 \cong \text{Var}(X_i) = \sigma^2 \Delta T.$$

- Therefore,

$$\sigma \cong \sqrt{\frac{1}{\Delta T} s_n^2}.$$

- We can use a spreadsheet to do the calculations. In **Excel** and **Google** Sheets, `VAR()` computes the sample variance of a set of data points .

(b) Use the estimated volatility to find the price of a 6−month 80−strike European call on the stock. Assume the current time is day 35 and the constant continuously compounded interest rate is 5%.

5. **Problem Description:** Suppose we have different options of the following underlying asset, which is stocks of Apple Inc. (AAPL). The following option data was obtained on March 15, 2018 and all these options mature on September 21, 2018. We list out the options as follows:

Call Strike	Price	Put Strike	Price
C165	20.50	P165	6.48
C170	17.30	P170	8.18
C175	14.33	P175	10.18
C180	11.70	P180	12.53
C185	9.35	P185	15.20
C190	7.40	P190	18.20
C195	5.78	P195	21.50

Table 1: Prices of Apple Inc. options which mature on September 21, 2018.

Question: Do the following:

(a) Use the least squares method to compute the present value of the forward price PVF and the discount factor $disc$ corresponding to the AAPL options.

(b) Compute the implied volatilities of the AAPL options using Newton's method. The algorithm is terminated when two consecutive approximations in Newton's method are within 10^{-6}. It is advisable to use any computing software for finding the implied volatilities of the AAPL options. How do the implied volatilities of calls and puts with the same strike compare to each other? In other words, based on the put-call parity, verify whether the theoretical values of the implied volatilities of calls and puts with the same strike are equal or not.

6. (Optional) The sensitivity of the `vega` of a portfolio with respect to volatility and to the price of the underlying asset are often essential to estimate, i.e., for pricing volatility swaps. These two Greeks are called `volga` and `vanna` and defined as follows:

$$\texttt{volga}(V) = \frac{\partial \texttt{vega}(V)}{\partial \sigma} \qquad \text{and} \qquad \texttt{vanna}(V) = \frac{\partial \texttt{vega}(V)}{\partial S}$$

The name `volga` is short for volatility gamma. Also, `vanna` can be interpreted as the rate of change of the Delta with respect to the volatility of the underlying asset, i.e.,

$$\texttt{vanna}(V) = \frac{\partial \Delta(V)}{\partial \sigma}.$$

(a) Compute the `volga` and `vanna`for a plain vanilla European call option on an asset paying dividends continuously at the rate q.

(b) Use the put-call parity to compute the `volga` and `vanna` for a plain vanilla European put option.

7. (Optional) Consider a put option with strike 55 and maturity 4 months on a non-dividend paying asset with spot price 60 which follows a lognormal model with drift $\mu = 0.1$and volatility $\sigma = 0.3$. Assume that the risk-free rate is constant and equal to 0.05.

- Find the probability that the put will expire in the money.
- Find the risk-neutral probability that the put will expire in the money.
- Compute $N(-d_2)$.

8. (Optional) Use risk-neutral (RN) pricing to price a supershare, i.e., an option that pays $\max(S(T) - K, 0))^2$ at the maturity of the option. In other words, compute

$$V(0) = e^{-rT} E_{RN}[(\max(S(T) - K, 0))^2],$$

where the expected value is computed with respect to the risk-neutral distribution of the price $S(T)$ of the underlying asset at maturity T, which is assumed to follow a lognormal process with drift r and volatility σ. Assume that the underlying asset pays no dividends, i.e., $q = 0$.