THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 18 More Real Integrals

18.1 Integrands having Branches

As we know, there is a new concept about functions in complex, that is, the concept of branches. A real function which has clear definition may become a function with branches in complex. Typical examples are $\ln x$ or x^r where $r \in \mathbb{R}$. This creates some troubles, but surprising also benefits.

18.1.1 Choose an Indented Contour

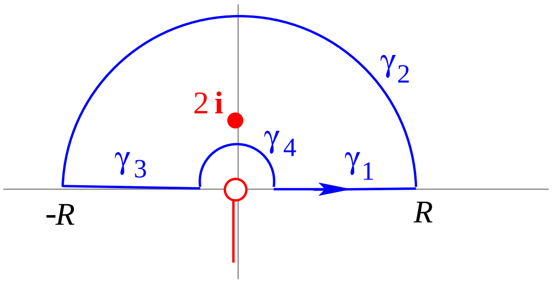
EXAMPLE 18.1. To evaluate $\displaystyle\int_0^\infty \frac{\ln x\,dx}{(x^2+4)^2}$. The natural complex function to consider is

$$f(z) = \frac{\text{A branch of } \log z}{(z^2+4)^2}.$$

Which branch of $\log z$ should we choose? Although there are many choices, we still need to choose it carefully. Of course, we would like to choose a convenient one to simplify the calculation. However, the choice must be compatible with the contour. Here are the key points.

- First, $\ln(x)$ and any branch of $\log z$ are not defined at the origin, we have to avoid the origin.
- Second, to get the result, we need the straight line γ_1 along the $\mathbb{R}$ from $\delta > 0$ to $R > 0$; then take limit $\delta \to 0$ and $R \to \infty$.
- Observe the integrand, besides $\ln x$, the remaining part $\dfrac{1}{(x^2+4)^2}$ is an even function, so we will use the straight line from $-R$ to $-\delta$. (Compare this step with the one in the next exercise).

With the above considerations, we will choose the contour $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ as shown below.



Moreover, we will take $\text{Log}_{-\pi/2}(z) = \ln|z| + \mathbf{i}\,\text{Arg}_{-\pi/2}(z)$, where $\text{Arg}_{-\pi/2}(z) \in \left(\dfrac{-\pi}{2}, \dfrac{3\pi}{2}\right)$.

First, since $2\mathbf{i}$ is a pole of order 2, the contour integral is given by

$$\int_\Gamma \frac{\operatorname{Log}_{-\pi/2}(z)}{(z+2\mathbf{i})^2(z-2\mathbf{i})^2}\,dz = 2\pi\mathbf{i}\lim_{z\to\mathbf{i}}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\operatorname{Log}_{-\pi/2}(z)}{(z+2\mathbf{i})^2}\right]$$
$$= 2\pi\mathbf{i}\lim_{z\to\mathbf{i}}\left[\frac{1/z}{(z+2\mathbf{i})^2}-\frac{2(\ln|2\mathbf{i}|+\mathbf{i}\pi/2)}{(z+\mathbf{i})^3}\right] = \frac{\pi(\ln 2-1)}{16}+\frac{\pi^2\mathbf{i}}{32}\,.$$

Second, similar as the methods learned before, and observe that $\operatorname{Arg}_{-\pi/2}(Re^{\mathbf{i}t}) = t \le \pi$,

$$\left|\int_{\gamma_2} f\right| \le \int_0^\pi \frac{\left|\ln(Re^{\mathbf{i}t})\right| + \left|\mathbf{i}\operatorname{Arg}_{-\pi/2}(Re^{\mathbf{i}t})\right|}{(R^2-4)^2}\left|R\mathbf{i}e^{\mathbf{i}t}\right|\,dt = \int_0^\pi \frac{\ln R+|t|}{(R^2-4)^2}\,R\,dt$$
$$\le \frac{\pi\,(\ln R+\pi)\,R}{(R^2-4)^2} \longrightarrow 0\,, \quad \text{as } R\to\infty.$$

Third, on the arc $-\gamma_4$, we have $z(t)=\delta e^{\mathbf{i}t}$ for $t\in[0,\pi]$ and $\operatorname{Arg}_{-\pi/2}(\delta e^{\mathbf{i}t}) = t\le\pi$. Thus,

$$\left|\int_{\gamma_4} f\right| \le \int_0^\pi \frac{\left|\ln(\delta e^{\mathbf{i}t})\right| + \left|\mathbf{i}\operatorname{Arg}_{-\pi/2}(\delta e^{\mathbf{i}t})\right|}{(4-\delta^2)^2}\left|\delta\mathbf{i}e^{\mathbf{i}t}\right|\,dt$$
$$\le \frac{\pi\,(\ln\delta+\pi)\,\delta}{(4-\delta^2)^2} \longrightarrow 0\,, \quad \text{as } \delta\to 0.$$

Fourth, it is easy to see that $\displaystyle\int_{\gamma_1} f(z)\,dz \longrightarrow \int_0^\infty \frac{\ln x\,dx}{(x^2+4)^2}$. It remains to work on γ_3.
On γ_3, we have $z(t)=t$ for $t\in[-R,-\delta]$; $\operatorname{Log}_{-\pi/2}(t) = \ln|t| + \mathbf{i}\operatorname{Arg}_{-\pi/2}(t) = \ln|t|+\mathbf{i}\pi$. So,

$$\int_{\gamma_3} f(z)\,dz = \int_{-R}^{-\delta}\frac{|t|+\mathbf{i}\pi}{(t^2+4)^2}\,dt = \int_\delta^R \frac{|t|+\mathbf{i}\pi}{(t^2+4)^2}\,dt \longrightarrow \int_0^\infty \frac{\ln x\,dx}{(x^2+4)^2} + \mathbf{i}\pi\int_0^\infty \frac{dx}{(x^2+4)^2}\,.$$

Summarizing the above, we get

$$2\int_0^\infty \frac{\ln x\,dx}{(x^2+4)^2} + \mathbf{i}\pi\int_0^\infty\frac{dx}{(x^2+4)^2} = \frac{\pi(\ln 2-1)}{16}+\frac{\pi^2\mathbf{i}}{32}\,.$$

It follows from comparing real and imaginary parts that

$$\int_0^\infty \frac{\ln x\,dx}{(x^2+4)^2} = \frac{\pi(\ln 2-1)}{32} \qquad \text{and} \qquad \int_0^\infty \frac{dx}{(x^2+4)^2} = \frac{\mathbf{i}}{32}\,.$$

Exercise 18.2. Convince yourself that if $\operatorname{Log}_\alpha$, i.e., $\operatorname{Arg}_\alpha(z)\in(\alpha,\alpha+2\pi)$ instead, as long the branch cut is away from the contour Γ, the results of the two integrals will be the same (but some of the steps may be different).

Exercise 18.3. Evaluate $\displaystyle\int_0^\infty \frac{\ln x\,dx}{(x^3+4)^2}$, in which the denominator of the integrand is slightly changed. Explain why the contour Γ above does not work. Instead, one should take γ_3 from $Re^{2\pi\mathbf{i}/3}$ to $\delta e^{2\pi\mathbf{i}/3}$.

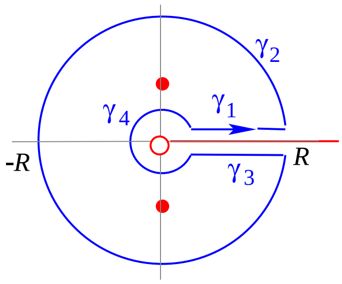
The above example and exercise demonstrate the following fact. Let f be a function that involves a branch. When it is restricted on suitable paths (γ_1 and γ_3 above), it mostly gives the real integrand with slight variations. In the way, the variation seems to give us trouble, but instead it makes the calculation work. This motivates the next method.

18.1.2 Along a Branch Cut

EXAMPLE 18.4. Let us try to work on the same integral $\displaystyle\int_0^\infty \frac{\ln x\,dx}{(x^2+4)^2}$ but we insist to use

$$g(z) = \frac{\operatorname{Log}_0 z}{(z^2+4)^2} = \frac{\ln|z| + \mathbf{i}\operatorname{Arg}_0(z)}{(z^2+4)^2}, \qquad \text{that is, the branch with } \operatorname{Arg}_0(z) \in (0, 2\pi).$$

For the chosen branch of logarithm, the cut is along the positive real axis. We may try the contour shown in the picture.



The line γ_1 is given by $t + \mathbf{i}\varepsilon$ for $t \in [\delta, R]$ and γ_3 is $R - t + \delta - \mathbf{i}\varepsilon$ for $t \in [\delta, R]$. The circles γ_2 and γ_4 are having radii R and δ respectively. Obviously, at the end, we will take limit $\delta \to 0$, $\varepsilon \to 0$, and $R \to \infty$.

Similar to previous calculations in Example 18.1, we have the estimates that

$$\left|\int_{\gamma_2} g(z)\,dz\right| \le \frac{2\pi R(\ln R + 2\pi)}{(R^2-4)^2} \qquad \text{and} \qquad \left|\int_{\gamma_4} g(z)\,dz\right| \le \frac{2\pi\delta(\ln\delta + 2\pi)}{(4-\delta^2)^2}\,.$$

These two integrals approach to 0 as $R \to \infty$ and $\delta \to 0$. Moreover, as $\delta, \varepsilon \to 0$ and $R \to \infty$,

$$\int_{\gamma_1} g(z)\,dz \longrightarrow \int_0^\infty \frac{\ln x\,dx}{(x^2+4)^2}\,.$$

On γ_3, we have $z(t) = R - t + \delta - \mathbf{i}\varepsilon$ for $t \in [\delta, R]$. Then $\operatorname{Log}_0 z(t) = \ln|z(t)| + \mathbf{i}\operatorname{Arg}_0(z(t))$, where $z(t) \to t$ and $\operatorname{Arg}_0(z(t)) \to 2\pi$ as $\varepsilon \to 0$. Thus,

$$\int_{\gamma_3} g(z)\,dz = \int_\delta^R \frac{\ln|z(t)| + \mathbf{i}\operatorname{Arg}_0(z(t))}{(z(t)^2+4)^2}\,(-dt) \longrightarrow \int_0^\infty \frac{-\ln x\,dx}{(x^2+4)^2} + \int_0^\infty \frac{-2\pi\mathbf{i}\,dx}{(x^2+4)^2}\,.$$

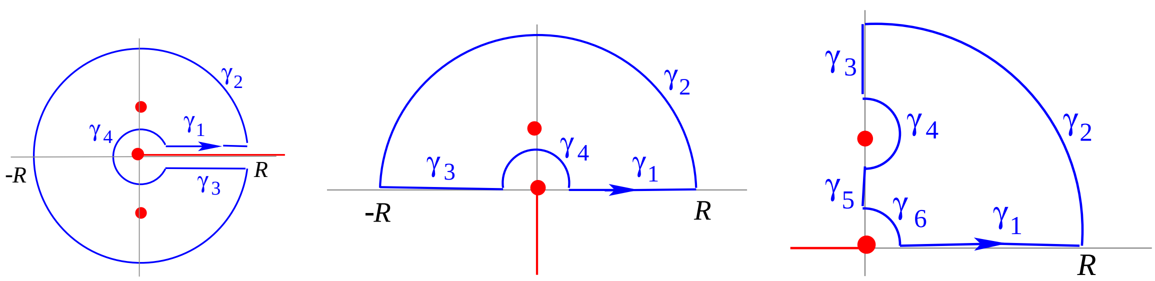
Thus, this contour will not give us what we want because the desired integral cancels out in

$$\int_{\gamma_1} g(z)\,dz + \int_{\gamma_3} g(z)\,dz \longrightarrow -2\pi\mathbf{i}\int_0^\infty \frac{dx}{(x^2+4)^2}\,.$$

EXERCISE 18.5. Somebody suggests that $\displaystyle\int_\Gamma \frac{(\operatorname{Log}_0 z)^2}{(z^2+4)^2}\,dz$, where Γ is the branch cut above, may give us the answer. Try this method.

18.1.3 A Tale of Three Methods

Let us evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+4)}$ by working on the contours Γ_a, Γ_b, and Γ_c with the branch cuts shown respectively from left to right below.



First, take the complex function $f(z) = \dfrac{1}{z^{1/2}(z^2+4)}$, which has singularities at 0, $2\mathbf{i}$, and $-2\mathbf{i}$. It also involves a branch of

$$z^{1/2} = e^{\frac{1}{2}\operatorname{Log}_\alpha z} = \exp\left(\frac{1}{2}\ln|z| + \frac{\mathbf{i}}{2}\operatorname{Arg}_\alpha z\right) = \sqrt{|z|}\exp\left(\frac{\mathbf{i}}{2}\operatorname{Arg}_\alpha z\right), \qquad \text{for suitable } \alpha.$$

We will take $\alpha = 0, \dfrac{3\pi}{2}$, and $-\pi$ respectively for Γ_a, Γ_b, and Γ_c.

EXAMPLE 18.6. For Γ_a and the branch cut at $\alpha = 0$, $\operatorname{Arg}_0(2\mathbf{i}) = \pi/2$ and $\operatorname{Arg}_0(-2\mathbf{i}) = 3\pi/2$. Therefore,

$$\begin{aligned}
(2\mathbf{i})^{1/2} &= e^{\frac{1}{2}\ln 2}\cdot e^{\frac{\mathbf{i}}{2}(\pi/2)} = \sqrt{2}\,e^{\pi\mathbf{i}/4} = 1+\mathbf{i}\,,\\
(-2\mathbf{i})^{1/2} &= e^{\frac{1}{2}\ln 2}\cdot e^{\frac{\mathbf{i}}{2}(3\pi/2)} = \sqrt{2}\,e^{3\pi\mathbf{i}/4} = -1+\mathbf{i}\,,\\
\operatorname{Res}(f, 2\mathbf{i}) &= \frac{1}{\sqrt{2}\,e^{\pi\mathbf{i}/4}(2\mathbf{i}+2\mathbf{i})} = \frac{-\mathbf{i}}{4\sqrt{2}}e^{-\pi\mathbf{i}/4} = \frac{-\mathbf{i}}{8}(1-\mathbf{i})\,,\\
\operatorname{Res}(f, -2\mathbf{i}) &= \frac{1}{\sqrt{2}\,e^{3\pi\mathbf{i}/4}(-2\mathbf{i}-2\mathbf{i})} = \frac{\mathbf{i}}{4\sqrt{2}}e^{-3\pi\mathbf{i}/4} = \frac{\mathbf{i}}{8}(-1-\mathbf{i})\,.
\end{aligned}$$

By Residue Theorem,

$$\int_{\Gamma_a} f(z)\,dz = 2\pi\mathbf{i}\cdot\frac{-\mathbf{i}}{8}\left[(1-\mathbf{i}) - (-1-\mathbf{i})\right] = \frac{\pi}{2}\,.$$

On γ_2, we may compare with the full circle C_R of radius R,

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_{C_R} |f(z)\,dz| \le \frac{2\pi R}{\sqrt{R}(R^2-4)} \longrightarrow 0\,.$$

Similarly, $\gamma_4 \subset C_\delta$, where C_δ is the circle with radius δ, and

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \int_{C_\delta} |f(z)\,dz| \le \frac{2\pi\delta}{\sqrt{\delta}(4-\delta^2)} \longrightarrow 0\,.$$

On γ_1, $z(t) = t + \mathbf{i}\varepsilon$, we have $\operatorname{Arg}_0 z(t) \to 0$ and $z(t)^{1/2} \to \sqrt{t}$ as $\varepsilon \to 0$. Thus,

$$\int_{\gamma_1} f(z)\,dz \longrightarrow \int_0^\infty \frac{dt}{\sqrt{t}(t^2+4)}\,.$$

On $-\gamma_3$, $z(t) = t - \mathbf{i}\varepsilon$. As $\varepsilon \to 0$, we have $\operatorname{Arg}_0 z(t) \to 2\pi$ and $z(t)^{1/2} \to -\sqrt{t}$. Therefore,

$$\int_{\gamma_3} f(z)\,dz \longrightarrow -\int_0^\infty \frac{dt}{-\sqrt{t}(t^2+4)} = \int_0^\infty \frac{dt}{\sqrt{t}(t^2+4)}\,.$$

To summarize, we have

$$2\int_0^\infty \frac{dx}{\sqrt{x}(x^2+4)} = \frac{\pi}{2}\,.$$

EXAMPLE 18.7. For the second contour Γ_b, we deliberately use $\alpha = 3\pi/2$ instead of $-\pi/2$ to illustrate how things will nicely cancel out. Here $3\pi/2 < \operatorname{Arg}_{3\pi/2}(z) < 7\pi/2$, then

$$\operatorname{Arg}_{3\pi/2}(2\mathbf{i}) = \sqrt{2}e^{5\pi\mathbf{i}/4} = -(1+\mathbf{i}) \qquad \text{and} \qquad \operatorname{Res}(f, 2\mathbf{i}) = \frac{\mathbf{i}}{8}(1-\mathbf{i})\,.$$

There is an additional *negative* when compared with the calculation in the cut of Γ_a. Nevertheless, we will see that things will work out fine. The estimates on γ_2 and γ_4 are beyond doubt and they go to zero. We only need to consider the situation along the real axis, i.e., γ_1 and γ_3.

On γ_1, $z(t) = t$ and $\mathrm{Arg}_{3\pi/2}(t) = 2\pi$. So, $z(t)^{1/2} = -\sqrt{t}$ and

$$\int_{\gamma_1} f(z)\,dz \longrightarrow \int_0^\infty \frac{dt}{-\sqrt{t}(t^2+4)} = -\int_0^\infty \frac{dx}{\sqrt{x}(x^2+4)}\,.$$

On $-\gamma_3$, $z(t) = -t$ and $\mathrm{Arg}_{3\pi/2}(-t) = 3\pi$, which leads to $z(t)^{1/2} = -\mathbf{i}\sqrt{t}$. Therefore,

$$\int_{\gamma_3} f(z)\,dz \longrightarrow \int_0^\infty \frac{dt}{-\mathbf{i}\sqrt{t}(t^2+4)} = \mathbf{i}\int_0^\infty \frac{dx}{\sqrt{x}(x^2+4)}\,.$$

From above, we already have calculated the residue at $2\mathbf{i}$ (note that $-2\mathbf{i}$ is outside Γ_b). Thus,

$$2\pi\mathbf{i}\cdot\frac{\mathbf{i}}{8}(1-\mathbf{i}) = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz \longrightarrow (-1+\mathbf{i})\int_0^\infty \frac{dx}{\sqrt{x}(x^2+4)}\,,$$

which gives the same answer $\pi/4$.

EXERCISE 18.8. Find out whether the contour Γ_c is helpful to get the answer.