THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 15 Power Series

15.1 Taylor Series

It is very easy to confuse the two concept of Taylor Expansion (which is finite) and Taylor Series (which is infinite). It is because most of the functions that we come across in high school are so nice. For these nice functions, the two concepts coincide.

15.1.1 Finite Expansion

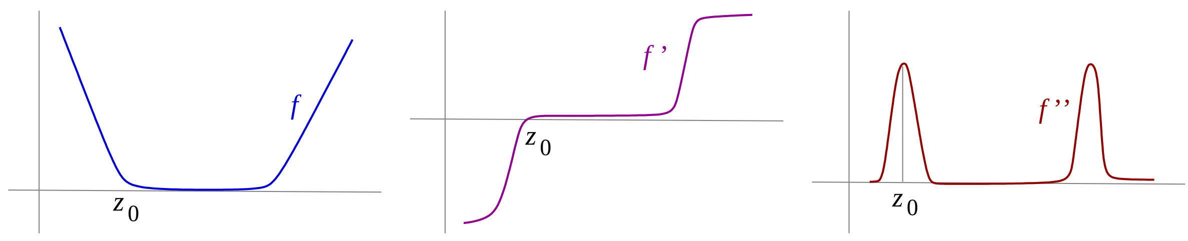
Let f be a function (real or complex variable) that has $(p+1)^{\text{st}}$ order of derivatives in a small ball $B(z_0,\eta)$. What can we say about its Taylor Expansion at z_0?

There exists $\boldsymbol{\delta}>0$ such that for $z\in B(z_0,\delta)$, there is $\boldsymbol{\zeta}$ satisfying $|\zeta-z_0|<|z-z_0|$,

$$f(z)=f(z_0)+\sum_{n=1}^{p}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\frac{f^{(p+1)}(\zeta)}{(p+1)!}(z-z_0)^{p+1}\,.$$

The crucial information in this statement are $\boldsymbol{\delta}$ and $\boldsymbol{\zeta}$, where not only do both depend on f and z_0 but also on p.

If the function f is C^∞, then we have for all $p\in\mathbb{N}$, there exist $\delta>0$ such that for $z\in B(z_0,\delta)$, there is ζ satisfying .... Although p can be arbitrarily large, it may occur that as $p\nearrow\infty$, $\delta\searrow 0$ or $\left|f^{(p+1)}(\zeta)\right|\to\infty$. In other words, when the right-hand side series is getting longer and longer, the equality may hold only for $z=z_0$ or the last error term may get too large. The following pictures show the example of a 1-variable C^∞ function f (left) and its derivatives f' (middle) and f'' (right).



In this example of $p=1$, one sees that near z_0, both $f(z_0),f'(z_0)\approx 0$ but $f''(\zeta)$ is large. However, this “weird” situation will not occur for analytic functions.

15.1.2 Infinite Series

Let f be a complex function analytic on $B(z_0,\eta)$. We hope to establish an expression of $f(z)$ in terms of an infinite series (called Taylor Series), i.e., on the ball $B(z_0,\eta)$ with same radius,

$$f(z)=f(z_0)+\sum_{n=1}^{\infty}a_n\,(z-z_0)^n\qquad\text{where}\quad a_n=\frac{f^{(n)}(z_0)}{n!}\,.$$

There are some technical conditions which will be seen from the proof. However, at this moment, it is beneficial for us to focus on the main ideas first.

It would be good to compare this expression for an analytic function with the previous one for a C^∞ function. What is the ***meaning*** of the equality of the infinite series? It actually means that the infinite series on the right hand side converges (indeed absolutely converges) for all $z \in B(z_0, \eta)$ to the same value as $f(z)$. Having the absolute convergences, we have a number of benefits, for examples,

- We can re-arrange infinitely many terms of the Taylor Series and the result is still $f(z)$;
- We can perform $+$, $-$, $\times$, $\div$ to the series similar to the operation of polynomials;
- We can differentiate or integrate the series term by term and the equality still holds.

In addition to the above, since the Taylor series is a power series of $(z - z_0)$, there are other helpful information. By Ratio Test or Root Test, one has the following conclusions. Let

$$\frac{1}{R} = \limsup_{n\to\infty} |a_{n+1}| \,/\, |a_n| = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \neq 0 \qquad \text{or} \quad R = \infty\,.$$

- If $|z - z_0| < R$, then the Taylor series converges absolutely; and
- if $|z - z_0| > R$, then the Taylor series diverges.

We say that the Taylor series has *radius of convergence* R and the *domain of convergence* is the ball $B(z_0, R)$ or the whole complex plane $\mathbb{C}$ when $R = \infty$.

Key Idea of Taylor Series. Let C be a circle with center z_0 and radius $r < \eta$. Then for any $z \in B(z_0, r)$, by Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)}{(\zeta - z)}\, d\zeta\,. \tag{$*$}$$

In order to express $f(z)$ as a power series of $(z - z_0)$, let us try to work on the integrand,

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0} \cdot \left[\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \right] = \sum_{n=0}^{\infty} \frac{1}{(\zeta - z_0)^{n+1}} \cdot (z - z_0)^n\,.
\end{aligned}$$

Substituting this into the integral, we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi\mathbf{i}} \int_C \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \cdot (z - z_0)^n\, d\zeta \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \right] \cdot (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n\,.
\end{aligned}$$

Apply the Cauchy Integral Formula again, we obtain the desired result that $a_n = \dfrac{f^{(n)}(z_0)}{n!}$. However, the above argument relies heavily on interchanging the infinite summation sign and the integral sign. This should be done carefully by analysis. The technical conditions and conclusion will be seen in the process of analysis. Indeed, for large N,

$$\frac{1}{\zeta - z} = \sum_{n=0}^{N} \frac{1}{(\zeta - z_0)^{n+1}} \cdot (z - z_0)^n + \left(\frac{z - z_0}{\zeta - z_0}\right)^{N+1} \cdot \frac{1}{\zeta - z}\,.$$

Substituting this into the equation $(*)$ for $f(z)$ above, and applying the Cauchy Integral Formula to $f^{(n)}(z_0)$, one has

$$f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \varepsilon\,, \qquad \text{where} \quad \varepsilon = \frac{(z - z_0)^{N+1}}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{N+1}\,(\zeta - z)}\,.$$

Taking absolute values, and letting $M_r = \sup\{\,\zeta \in \mathbb{C} : \ |\zeta - z_0| = r\,\}$, we have

$$|\varepsilon| \leq \frac{1}{2\pi} \cdot \frac{|z - z_0|^{N+1}\; M_r}{r^{N+1}} \cdot \left|\int_C \frac{d\zeta}{\zeta - z}\right| \leq M_r \left(\frac{|z - z_0|}{r}\right)^{N+1}\,,$$

in which, $|z - z_0| < r$ and so $|\varepsilon|$ can be made arbitrarily small by taking large N. $\square$

Note that in the above discussion, we worked on $|\varepsilon|$ and so absolute convergence of the series can be concluded. On the other hand, the argument depends on the upper bound M_r, which depends on r. There are two versions of theorem statement that present the similar results.

THEOREM 15.1 (Taylor Series on compact subsets). *Let f be a complex function analytic on $B(z_0, \eta)$. Then for all compact subset $K \subset B(z_0, \eta)$ and $z \in K$, the Taylor Series converges absolutely.*

THEOREM 15.2 (Power Series for bounded functions). *Let f be a bounded complex function on $B(z_0, \eta)$. Then for all $z \in B(z_0, \eta)$, the power series converges absolutely, namely,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \textit{where} \quad a_n = \frac{1}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}\,.$$

Proof for both Theorems. For any compact subset K, we can choose a radius $r < \eta$ such that $K \subset \overline{B(z_0, r)}$ and so the above M_r is independent of K. For the second statement, if f is bounded, let $M = \sup|f|$ on $B(z_0, \eta)$, then $M_r \leq M$. $\square$

EXERCISE 15.3. Which theorem of the above is applicable to an entire function?

Remark. Note that for Theorem 15.1 above, even if the compact set K or the radius $r < \eta$ is changed, the Taylor Series is still the same. Only the absolute convergence or uniform convergence depends on K or r.

15.1.3 More about series

It should be noted that most of the time, we use some other working procedures to find the Taylor series of a function.

EXAMPLE 15.4. Let $f(z) = e^z - 1$ and take $z_0 = 0$. It is easy to work out that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots\,.$$

Since we are allowed to re-arrange infinitely many terms and group terms, we have

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z + \frac{z^2}{2!} + \cdots \quad = z\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right).$$

In fact, the absolute convergence allows us to *do things as finite* on the suitable domain, for example, this will give us the Taylor Series for $\sin z$ at $z_0 = \pi/2$,

$$\sin z = \cos\left(z - \frac{\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(z - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2!}\left(z - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(z - \frac{\pi}{2}\right)^4 + \cdots.$$

Example 15.5. Consider the function $g(z) = \dfrac{\sin z}{z}$ which is defined and analytic on $\mathbb{C} \setminus \{0\}$. According to the theory above, we are not sure if there is a Taylor series for g at $z_0 = 0$. However, for each $z \in \mathbb{C}$,

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots.$$

Thus on any compact $K \subset \mathbb{C} \setminus \{0\}$ and $z \in K$, we have absolute convergence of

$$g(z) = \frac{\sin z}{z} = \frac{1}{z} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots.$$

The interesting fact is that the infinite series on the right hand side actually is defined for $z = 0$ and converges on $\mathbb{C}$. Therefore, there is a natural extension of g to the function

$$\hat{g}(z) = \begin{cases} \dfrac{\sin z}{z} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

This function $\hat{g}$ is more than just continuous at $z = 0$, but indeed is analytic on $\mathbb{C}$ and its Taylor Series is the one above.

Example 15.6. Consider the same function $g(z) = \dfrac{\sin z}{z}$ at $z_0 = \pi/2$. Then

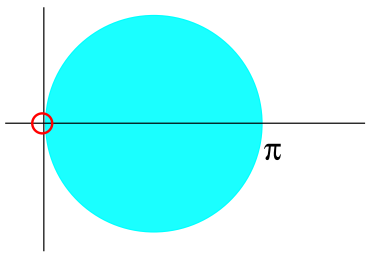
$$\sin z = \cos\left(z - \frac{\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(z - \frac{\pi}{2}\right)^{2k},$$

$$\frac{1}{z} = \frac{1}{\frac{\pi}{2} + \left(z - \frac{\pi}{2}\right)} = \frac{2}{\pi} \cdot \sum_{\ell=0}^{\infty} \left(\frac{-2}{\pi}\right)^{\ell} \cdot \left(z - \frac{\pi}{2}\right)^{\ell}, \qquad \text{for } \left|z - \frac{\pi}{2}\right| < \frac{\pi}{2}.$$

Taking the product of these two series, we have

$$\frac{\sin z}{z} = \frac{2}{\pi} \cdot \sum_{n=0}^{\infty} \left[\sum_{2k+\ell=n} \frac{(-1)^{k+\ell}}{(2k)!}\left(\frac{2}{\pi}\right)^{\ell}\right]\left(z - \frac{\pi}{2}\right)^n, \qquad \text{for } z \in B\left(\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The domain of convergence of the series is shown in the figure. Note that, unlike the above, it is not easy to conclude that this series converges at $z = 0$ although we expect that g can be extended to an entire function.



Exercise 15.7. Try substituting $z = 0$ into the above series.

15.2 Bad Terms Occur

As we have seen before, we often have to deal with functions that are not totally analytic. They may have "singularities" at some points, e.g., a zero on the denominator.

EXAMPLE 15.8. Let $f(z) = \dfrac{\sin z}{z - \frac{\pi}{2}}$. This function is not analytic at $z_0 = \dfrac{\pi}{2}$. Nevertheless, we still are able to express it into infinite series

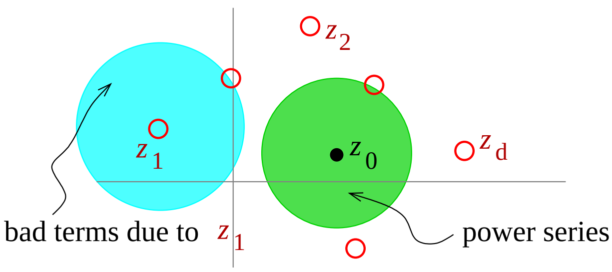
$$\begin{aligned}
\sin z &= \cos\left(z - \frac{\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(z - \frac{\pi}{2}\right)^{2k}, \\
f(z) &= \frac{1}{z - \frac{\pi}{2}} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(z - \frac{\pi}{2}\right)^{2k} \\
&= \frac{1}{z - \frac{\pi}{2}} - \frac{1}{2!}\left(z - \frac{\pi}{2}\right) + \frac{1}{4!}\left(z - \frac{\pi}{2}\right)^3 - \frac{1}{6!}\left(z - \frac{\pi}{2}\right)^5 + \cdots\cdot . \qquad (\dagger)
\end{aligned}$$

Note that the series ($\dagger$) is defined and converges absolutely on the ball with center $z_0 = \dfrac{\pi}{2}$ and radius $\dfrac{\pi}{2}$. The first term is "bad" that makes the series unable to have a larger radius of convergence. However, inside the ball, we may comfortably use the series.

The above example shows a rather general situation of a function of the form

$$g(z) = \frac{\text{(analytic)}}{z - z_1} \qquad \text{or further} \qquad \frac{\text{(analytic)}}{\text{(polynomial)}} = \frac{\text{(analytic)}}{(z - z_1)^{p_1} \cdot \cdots \cdot (z - z_d)^{p_d}}.$$

Suppose we would like to express in power series of $(z - z_0)$. If $z_0 \neq z_1, \ldots, z_d$, then $g(z)$ is analytic in a ball $B(z_0, \eta)$ for some $\eta > 0$ and so it has a Taylor Series absolutely convergent on $B(z_0, R)$. It turns out that $R = \min\{\,|z_0 - z_k| : \ k = 1, \ldots, d\,\}$, i.e., the shortest distance of z_0 from the singularities.



On the other hand, if we take the center to be one of the singularities, say z_1, then

$$g(z) = \frac{h(z)}{(z - z_1)^{p_1}} \qquad \text{where } h(z) = \frac{\text{(analytic)}}{(z - z_2)^{p_2} \cdots (z - z_d)^{p_d}}.$$

The function $h(z)$ is analytic in a small ball $B(z_1, \xi)$ and so it can be written as a power series in $(z - z_1)$. As consequence, $g(z)$ has a series expression with some "bad terms" of the form $(z - z_1)^{-p_1}$ up to $(z - z_1)^{-1}$.

After understanding a function with polynomial denominator, the next challenge is one with an analytic function as denominator. The situation is indeed similar, only there may be infinitely many singularities.

Example 15.9. Let $f(z) = \dfrac{1}{e^z - 1}$ for which the singularities are $2k\pi\mathbf{i}$ for $k \in \mathbb{Z}$. We will look at the series expression of $f(z)$ at the point $z_0 = 0$. The denominator has a power series

$$e^z - 1 = z \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right),$$

which converges absolutely on $\mathbb{C}$. Then

$$\frac{1}{e^z - 1} = \frac{1}{z} \cdot \frac{1}{g(z)} \qquad \text{where } g(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots.$$

Here, g is analytic and $g(0) = 1 \neq 0$. Therefore, on a small ball $B(0,\eta)$, the function $1/g(z)$ is analytic and it has a power series at $z_0 = 0$,

$$\frac{1}{g(z)} = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots.$$

The coefficients can be iteratively solved from the equation

$$1 = g(z) \sum_{k=0}^{\infty} a_k z^k = \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right) \cdot \left(a_0 + a_1 z + a_2 z^2 + \cdots\right).$$

That leads to $1 \cdot a_0 = 1$ and $\dfrac{1}{2!} \cdot a_0 + 1 \cdot a_1 = 0$ and so on. Thus, we have $a_0 = 1$, $a_1 = -1/2$, $a_2 = 1/12$, etc. As a result,

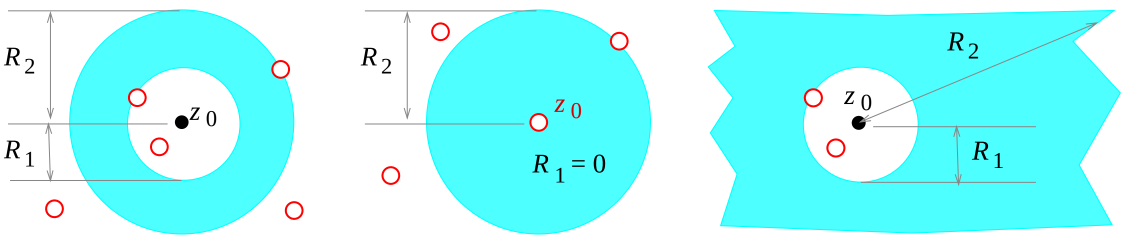
$$\frac{1}{e^z - 1} = \frac{1}{z} \cdot \frac{1}{g(z)} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{6!} + \frac{z^5}{6 \cdot 7!} + \cdots \quad \text{which converges absolutely on } \mathbb{C} \setminus \{0\}.$$

15.2.1 Laurent Series

In the previous section, we have shown some examples of functions having singularities. If z_0 is a singularity, the function still has a series expression of $(z - z_0)$ but there may be some negative powers. Such a series is called a *Laurent Series*.

Taylor series is about analytic functions; Laurent series is about "almost" analytic functions with singularities. Mathematically, let f be analytic on an annulus

$$A(z_0, R_1, R_2) :\stackrel{\text{def}}{=\!=} \{\, z \in \mathbb{C} : \; R_1 < |z - z_0| < R_2 \,\}.$$



The situation is shown in the pictures above, in which the red circles indicate points that the function f is not analytic. From this, you will see that being *analytic on an annulus* is truly an "almost analytic" condition. Also, at the center point z_0, f can be analytic or not.

Theorem 15.10 (Laurent Series). *Let f be analytic on an annulus $A(z_0, R_1, R_2)$. Then,*

$$f(z) = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} + \sum_{\ell=0}^{\infty} a_\ell (z-z_0)^\ell\,, \qquad \textit{for } z \in A(z_0, R_1, R_2)\,,$$

$$\textit{where} \qquad a_{-k} = \frac{1}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{-k+1}} \quad \textit{and} \quad a_\ell = \frac{1}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{\ell+1}}\,,$$

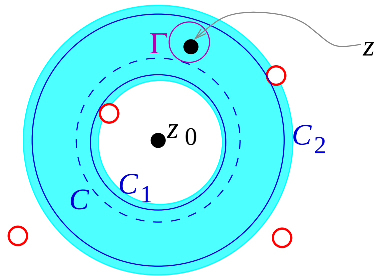
where C is any circle with center z_0 and radius between R_1, R_2.

Remark. As C is a circle inside the annulus $A(z_0, R_1, R_2)$, its inside contains the ball $B(z_0, R_1)$ which may contain singularity of f. Therefore

$$a_\ell = \frac{1}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{\ell+1}} \neq \frac{f^{(\ell)}(z_0)}{\ell!}\,.$$

On the other hand, if indeed f is analytic on $B(z_0, R_1)$, then the above becomes equality. Moreover, the coefficients $a_{-k} = 0$ because the integrand $f(\zeta)(z - z_0)^{k-1}$ becomes analytic. The series becomes a power series.

Proof of Laurent Series. The proof is very similar to the one of Taylor Series. Let $z \in A(z_0, R_1, R_2)$. Then, as shown in the picture below, we may find two circles with center z_0 where C_1 has radius $R_1 + \delta$, C_2 has radius $R_2 - \delta$, and also a simple closed contour $\Gamma \subset A(z_0, R_1, R_2)$ containing z in its inside. We also indicate any circle C with center z_0 in the annulus by a dotted circle.



First, by Cauchy Integral Formula applied on the function f, we have

$$f(z) = \frac{1}{2\pi\mathbf{i}} \int_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}\,.$$

Then, observe that the integrand $f(\zeta)/(\zeta - z)$ is analytic on the region *between* the contours C_1, C_2, Γ. We may then further to obtain

$$f(z) = \frac{1}{2\pi\mathbf{i}} \int_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi\mathbf{i}} \int_{C_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi\mathbf{i}} \int_{C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z}\,. \tag{15.10}$$

On the circle C_2, we will use exactly the same method as in Taylor Series to write,

$$\frac{1}{\zeta - z} = \sum_{\ell=0}^{N} \frac{(z-z_0)^\ell}{(\zeta - z_0)^{\ell+1}} + (\text{error})\,,$$

where $(\text{error}) \to 0$ as $N \to \infty$ because $|\zeta - z_0| > |z - z_0|$ for $z \in C_2$. This gives rise to the positive power terms and the expression for a_ℓ. The rigorous analysis about absolute (and uniform) convergence should be done as in the proof for the Taylor Series.

On the circle C_1, the situation is reversed. We write

$$\frac{1}{\zeta - z} = \frac{1}{z - z_0} \cdot \frac{-1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\sum_{k=0}^{N} \frac{(\zeta - z_0)^k}{(z - z_0)^{k+1}} + (\text{error})\,.$$

Now, on C_1, $|\zeta - z_0| < |z - z_0|$ and so (error) $\to 0$. Note that the negative sign in front of the summation cancels out the negative of the integral in Equation (15.10) above. Then by shifting the index k and rewriting the expression, we have

$$\frac{f(\zeta)}{\zeta - z} = -\sum_{k=1}^{N+1} \frac{f(\zeta)(\zeta - z_0)^{k-1}}{(z - z_0)^k} + (\text{error}) = -\sum_{k=1}^{N+1} \frac{f(\zeta)}{(\zeta - z_0)^{-k+1}} \cdot \frac{1}{(z - z_0)^k} + (\text{error})\,,$$

which clearly gives the desired result of a_{-k}. □

At this point, it is educational to illustrate the theory by a simple example. Let

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2} \qquad z \in \mathbb{C} \setminus \{1, 2\}\,.$$

For this single function f, there are actually seven typical series expression depending on the domain of convergence in concern.

Take $z_0 = 0$, as a typical case for $z_0 \neq 1, 2$. The function f is analytic on $B(z_0, 1)$ so it has a Taylor Series found by the following steps.

$$\begin{aligned}
\frac{1}{z-1} &= \frac{-1}{1-z} = -\sum_{\ell=0}^{\infty} z^\ell = -\left(1 + z + z^2 + z^3 + \cdots\right)\,, \quad |z| < 1 \\
\frac{1}{z-2} &= \frac{-1}{2} \cdot \frac{1}{1 - \frac{z}{2}} = \frac{-1}{2} \sum_{\ell=0}^{\infty} \frac{z^\ell}{2^\ell} = -\left(\frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \cdots\right)\,, \quad |z| < 2 \\
f(z) &= \frac{-1}{(z-1)(z-2)} = \sum_{\ell=0}^{\infty} \left(-1 + \frac{1}{2^{\ell+1}}\right) z^\ell = -\left(\frac{1}{2} + \frac{3z}{4} + \frac{7z}{8} + \frac{15z}{16} + \cdots\right)\,.
\end{aligned}$$

On the annulus $A(z_0, 1, 2)$, f is also analytic and so it has a Laurent Series. As $|z| < 2$, the second series above is still valid. Yet,

$$\begin{aligned}
\frac{1}{z-1} &= \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{z^k} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots\,, \quad |z| > 1 \\
f(z) &= \sum_{k=1}^{\infty} \frac{1}{z^k} + \sum_{\ell=0}^{\infty} \frac{z^\ell}{2^{\ell+1}} = \cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \cdots\,.
\end{aligned}$$

Finally, f is also analytic on $A(z_0, 2, \infty)$ and has a Laurent Series there. For $|z| > 2$, we have

$$\begin{aligned}
\frac{1}{z-2} &= \frac{1}{z} \cdot \frac{1}{1 - \frac{2}{z}} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{2^k}{z^k} = \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \cdots \\
f(z) &= \sum_{k=0}^{\infty} \frac{1 - 2^k}{z^{k+1}} = -\left(\cdots + \frac{15}{z^5} + \frac{7}{z^4} + \frac{3}{z^3} + \frac{1}{z^2}\right)\,.
\end{aligned}$$

Exercise 15.11. Find the two Laurent series expansions for f at each $z_0 = 1$ or $z_0 = 2$.