THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 9 Contour Integration

## 9.1 Contour Integral

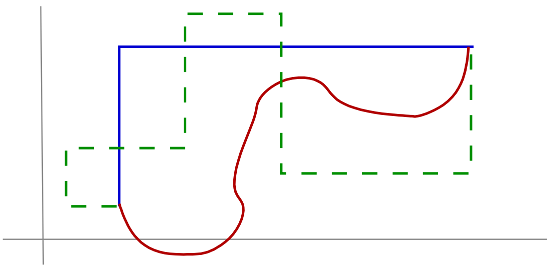
Previously, we have learned how to find the harmonic conjugate of a given harmonic function $u(x, y)$. In other words, we would like to find $v(x, y)$ such that $u+\mathbf{i}v$ defines an analytic function. The process is to use the Cauchy-Riemann Equations successively. For example, we start with

$$v_y = u_x = \quad \text{this is known as } u \text{ is given},$$

and obtain

$$v = \int u_x \, dy + \varphi(x) .$$

Then, we use $v_x = -u_y$ to get information about $\varphi'(x)$. Next, integrating wrt $x$ will get the result. Observing from both analytically and geometrically, we see that the process corresponds to an integration along the vertical and then horizontal line in the picture below.



Sometimes, in order to avoid holes in the domain $\Omega$ where $u$ is defined, we may need to take more vertical and horizontal lines (dotted in picture). Nevertheless, one may expect the general theory should be related to integration along a curve.

### 9.1.1 Contours and Integrals

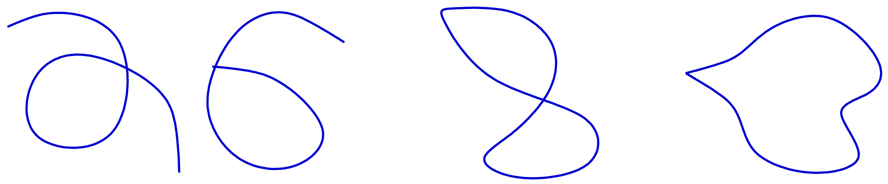
Let us start with the building block of a contour. A *smooth path* $\gamma$ in $\Omega \subset \mathbb{C}$ is a continuous function $\gamma : [a, b] \to \Omega$ from an interval with $\gamma(t) = x(t) + \mathbf{i}y(t)$ such that $x'(t)$ and $y'(t)$ exist and are continuous for all $t \in (a, b)$. The function $\gamma$ is called a *parametrization* of the path. It determines a "direction" for the path. We denote $\gamma'(t) = x'(t) + \mathbf{i}y'(t)$. The parametrization is *regular* if $\gamma'(t) \neq 0$ for all $t \in (a, b)$.

Example 9.1. Typical examples are

- Straight line joining $z_0, z_1 \in \mathbb{C}$: $t \in [0, 1] \mapsto (1 - t)z_0 + tz_1$.
- Circle with center $z_0$ and radius $R$: $t \in [0, 2\pi] \mapsto z_0 + Re^{\mathbf{i}t}$.

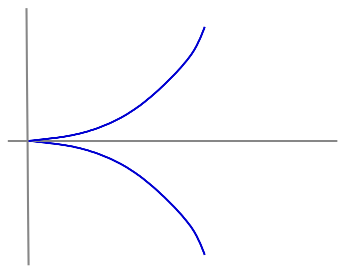
- Same circle but going clockwise and different end-points: $t \in [-\pi, \pi] \mapsto z_0 + Re^{-\mathbf{i}t}$.

A smooth arc is *closed* if the end-points are the same, i.e., $\gamma(a) = \gamma(b)$. It is *simple* if $\gamma$ is one-one on $[a, b)$ and $(a, b]$, i.e., no self-intersection except at the end points.

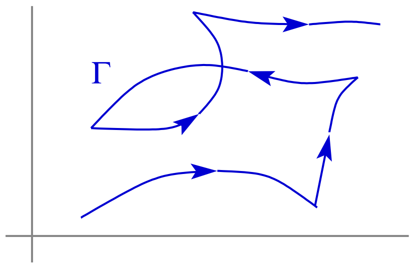


Not simple not closed Not simple Simple closed

Example 9.2. A curve with a sharp point may have a differentiable parametrization, but such parametrization is not regular. Try to give a differentiable parametrization for the following curve defined by $y^2 = x^3$.



A *contour* $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ in $\Omega \subset \mathbb{C}$ consists of finitely many smooth arcs $\gamma_1, \ldots, \gamma_n$ such that the terminal point of $\gamma_k$ is the starting point of $\gamma_{k+1}$ for $k = 1, \ldots, n-1$.




Let $\gamma : [a, b] \to \Omega \subset \mathbb{C}$ be a smooth arc and $f : \Omega \to \mathbb{C}$, i.e., the domain of $f$ contains the arc. The *contour integral* of a continuous function $f$ along $\gamma$ is denoted and defined by

$$\int_\gamma f \quad \text{or} \quad \int_\gamma f(z)\,dz :\stackrel{\text{def}}{=\!=} \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$
$$= \int_a^b f(x(t) + \mathbf{i}y(t))\,\big(x'(t) + \mathbf{i}y'(t)\big)\,dt$$
$$:\stackrel{\text{def}}{=\!=} \begin{pmatrix} \text{A real integral of} \\ \text{the real part wrt } t \end{pmatrix} + \mathbf{i} \begin{pmatrix} \text{A real integral of} \\ \text{the imaginary part wrt } t \end{pmatrix}$$

For a general contour $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, we define

$$\int_\Gamma f :\stackrel{\text{def}}{=\!=} \sum_{k=1}^{n} \int_{\gamma_k} f\,.$$

EXAMPLE 9.3. Let $f(z) = z+1$ and $\gamma$ be the straight line from $-1$ to $1+\mathbf{i}$, i.e., $\gamma(t) = 2t-1+\mathbf{i}t$ for $t \in [0,1]$. Then

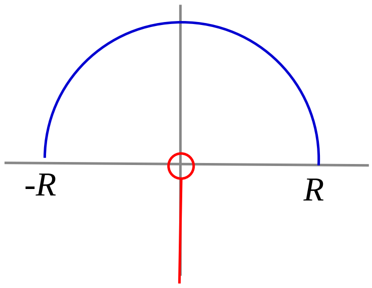
$$\int_\gamma f = \int_0^1 (2t - 1 + \mathbf{i}t + 1) \cdot (2+\mathbf{i})\, dt = \int_0^1 3t\, dt + \int_0^1 4t\mathbf{i}\, dt\,.$$

EXAMPLE 9.4. Let $f(z) = \overline{z}$ and $\gamma$ be the semi-circle with center 0, radius $R$, counter-clockwise from $-R\mathbf{i}$ to $R\mathbf{i}$. Then $\gamma(t) = Re^{\mathbf{i}t}$ for $t \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and

$$\int_\gamma f(z)\, dz = \int_{-\pi/2}^{\pi/2} \overline{\left(Re^{\mathbf{i}t}\right)} \cdot R\mathbf{i}e^{\mathbf{i}t}\, dt = \pi R^2 \mathbf{i}\,.$$

EXAMPLE 9.5. This is a case that commonly occurs, in which the integrand is a multi-valued functions. Let $f(z) = z^{1/2}$ and $\gamma$ be the semi-circle with center 0, radius $R$, counter-clockwise from $R$ to $-R$. First, we may parametrize the curve by $\gamma(t) = Re^{\mathbf{i}t}$ for $t \in [0,\pi]$.

Since $f(z) = z^{1/2} = e^{\frac{1}{2}\log z}$ is indeed a set, we have to choose a branch of it which is defined on $\gamma$. For this, let $\alpha = -\pi/2$ and take the branch $\exp\left(\frac{1}{2}\operatorname{Log}_\alpha z\right)$ on $\mathbb{C} \setminus H_\alpha$.




Then, $\operatorname{Log}_\alpha \gamma(t) = \operatorname{Log}_\alpha\left(Re^{\mathbf{i}t}\right) = \ln R + \mathbf{i}\operatorname{Arg}_\alpha\left(Re^{\mathbf{i}t}\right) = \ln R + \mathbf{i}t$. *Warning.* This calculation depends on $\alpha$ (the choice of branch of $\log z$) and also the parametrization. See the remark below for more discussion.

In our case, using the cut and the branch of $\log z$ at $\alpha = -\pi/2$, we have

$$\begin{aligned}\int_\gamma f &\overset{\text{def}}{=\!=} \int_0^\pi f\left(Re^{\mathbf{i}t}\right) \cdot R\mathbf{i}e^{\mathbf{i}t}\, dt = \quad \cdots\cdots \quad = R^{3/2}\int_0^\pi \mathbf{i}e^{3\mathbf{i}t/2}\, dt \\ &= R^{3/2}\left[\int_0^\pi \left(-\sin\frac{3t}{2}\right) dt \quad + \quad \mathbf{i}\int_0^\pi \cos\frac{3t}{2}\, dt\right] = \quad \cdots\cdots \quad = \frac{-2}{3}R^{3/2}(1+\mathbf{i})\,.\end{aligned}$$

*Remark.* In the calculation of $\operatorname{Log}_\alpha \gamma(t)$ above, there are several complications. First, at the same cut of the negative imaginary axis, there is a choice of $\alpha$. In the above, we chose $\alpha = -\pi/2$, then $\operatorname{Arg}_\alpha(Re^{\mathbf{i}t}) = t$. For instance, if $\alpha = 3\pi/2$ is chosen, as a set, we have the same branch cut at the negative imaginary axis; but $\operatorname{Arg}_\alpha\left(Re^{\mathbf{i}t}\right) = t + \pi$ because $t \in [0,\pi]$. The parameter $t$ also has an effect. For example, the same semi-circle may be parametrized by $z(t) = -R(\sin t + \mathbf{i}\cos t)$ with $t \in [-\pi/2, \pi/2]$. In such a case, if we take $\alpha = -\pi/2$, $\operatorname{Arg}_\alpha(Re^{\mathbf{i}t}) = t + \pi/2$. Or even worse, one may take the parametrization $z(t) = -t + \mathbf{i}\sqrt{R^2 - t^2}$ for $t \in [-R, 0]$. In this case, $\operatorname{Arg}_\alpha(Re^{\mathbf{i}t})$ is a complicated expression in terms of $t$.

EXERCISE 9.6. Try the above by taking another branch of $z^{1/2}$ or other parametrization.

### 9.1.2 Fundamental Theorem, True or Not??

From the calculations in all the above examples, we see a pattern and a natural question arise. In principle,

$$\int_\gamma f = \quad \cdots\cdots \quad = \int_a^b (\text{Something in } t)\, dt + \mathbf{i} \int_a^b (\text{Another stuff in } t)\, dt\,.$$

But, at the end, we inevitably used the Fundamental Theorem of Calculus to find the values of the integrals. Can we do the contour integration faster? First of all, it is not necessary to break it into real and imaginary parts before doing integration. In Example 9.5, we can simply write

$$\int_0^\pi \mathbf{i}e^{3\mathbf{i}t/2}\, dt = \frac{2}{3}e^{3\mathbf{i}t/2}\bigg|_0^\pi\,, \tag{$\star$}$$

because $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{2}{3}e^{3\mathbf{i}t/2}\right) = \mathbf{i}e^{3\mathbf{i}t/2}$. The Fundamental Theorem of Calculus for real integrals can be used because in both the left hand side and right hand side of the equation ($\star$) above, it only involves algebraic operations to break it to real and imaginary parts.

Exercise 9.7. Write a mathematical statement that corresponds to the general situation of equation ($\star$). This statement is very easy to prove, in fact, trivial.

More importantly, one would ask whether we can get the answer directly from the integrand $f(z)$ and the arc $\gamma$. It seems that it is possible for Examples 9.3 and 9.5, but not so obvious for 9.4, though in Example 9.5, there are four possibilities to pick the answer.

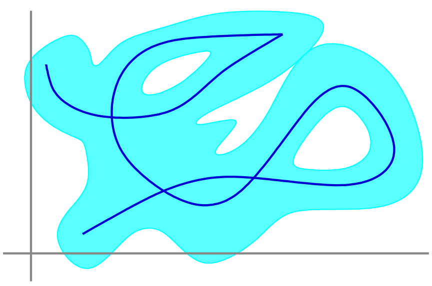
(9.3) $\displaystyle\int_\gamma f = \frac{z^2}{2} + z\bigg|_{-1}^{1+\mathbf{i}} = \frac{3}{2} + 2\mathbf{i};$

(9.5) $\displaystyle\int_\gamma f = \frac{2}{3}z^{3/2}\bigg|_R^{-R} = \frac{-2}{3}R^{3/2}\left[(1)^{3/2} - (-1)^{3/2}\right].$

Apparently, there is a certain version of Fundamental Theorem of Calculus if the integrand $f$ is good enough.

## 9.2 Antiderivatives

Given a function $f:\ \Omega \subset \mathbb{C} \to \mathbb{C}$ and a contour $\Gamma \subset \Omega$. A function $F:\ \Omega \to \mathbb{C}$ is called an *antiderivative* of $f$ in $\Omega$ if $F'(z) = f(z)$ for all $z \in \Omega$. Note that there is almost no requirement on the domain $\Omega$. It may have holes and only needs to contain the contour $\Gamma$.



THEOREM 9.8. *If $\Gamma$ is a contour in $\Omega$ from the point $z_0$ to $z_1$ and $f$ has an antiderivative $F$ in $\Omega$, then*

$$\int_\Gamma f = F(z_1) - F(z_0)\,.$$

The main idea of the proof is changing of variables in integration. Let $f = u + \mathbf{i}v$ and $\Gamma$ can be parametrized by $t \in [a,b] \mapsto z(t) = x(t) + \mathbf{i}y(t)$. In addition, we have $F = P + \mathbf{i}Q$ and $F' = f$. For simplicity, we only write $u$ instead of $u(x(t), y(t))$ for the real part of the composite $f \circ \Gamma$ and similarly for $v, P, Q$. Then

$$\begin{aligned}
\text{Real part of } \int_\Gamma f = \mathrm{Re}\left(\int_a^b f \circ \Gamma(t) \cdot \Gamma'(t)\,dt\right) &= \int_a^b \big(u \cdot x'(t) - v \cdot y'(t)\big)\,dt \\
= \int_a^b \big(P_x \cdot x'(t) - Q_x \cdot y'(t)\big)\,dt &= \int_a^b \big(P_x \cdot x'(t) + P_y \cdot y'(t)\big)\,dt \\
= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[P \circ \Gamma(t)\right]\,dt &= P \circ \Gamma(b) - P \circ \Gamma(a) = P(z_1) - P(z_0)\,.
\end{aligned}$$

The situation for the imaginary part is similar.

From this theorem, whenever there is an antiderivative of the integrand $f$ on an open set containing the contour, we can have something similar to Fundamental Theorem of Calculus. The contour integral only depends on the two end-points, but not the contour itself.

COROLLARY 9.9. *Let $f, F : \Omega \subset \mathbb{C} \to \mathbb{C}$ and $F'(z) = f(z)$ for all $z \in \Omega$.*

1. *If $\Gamma_1, \Gamma_2 \subset \Omega$ are two contours with the same end-points, then*

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f\,.$$

2. *If $\Gamma \subset \Omega$ is a closed contour, then* $\displaystyle\int_\Gamma f = 0.$

Let us *emphasize* that the domain $\Omega$ may have ***holes*** in this case and the contour $\Gamma$ may have self-intersections. There will be another similar but different theorem later.

Let us look at some examples to understand more.

EXAMPLE 9.10. Let $\gamma$ be the circle with center 0 and radius $R$ in counterclockwise direction, i.e., $\gamma(t) = Re^{\mathbf{i}t}$ for $t \in [0, 2\pi]$ and $f(z) = 1/z^2$. By direct calculation,

$$\int_\gamma f = \int_0^{2\pi} \frac{1}{R^2 e^{2\mathbf{i}t}} \cdot R\mathbf{i}e^{\mathbf{i}t}\,dt = \frac{\mathbf{i}}{R}\int_0^{2\pi} e^{-\mathbf{i}t}\,dt = \frac{\mathbf{i}}{R}\left(-\mathbf{i}e^{-\mathbf{i}t}\right)\Big|_0^{2\pi} = 0\,.$$

The result is consistent with Corollary 9.9 by taking $\Omega = \mathbb{C} \setminus \{0\}$.

EXAMPLE 9.11. What if we take the same circle but $f(z) = 1/z$ instead? Again, by direct calculation,

$$\int_\gamma f = \int_0^{2\pi} \frac{1}{Re^{\mathbf{i}t}} \cdot R\mathbf{i}e^{\mathbf{i}t}\,dt = \mathbf{i}\int_0^{2\pi} 1\,dt = 2\pi\mathbf{i} \neq 0\,.$$

Why is not Corollary 9.9 applicable to this integral? Also, observe that the answer does not depend on the radius of the circle.

As a matter of fact, we may use similar method as in Example 9.10 to conclude that

$$\int_\gamma z^n \, dz = \begin{cases} 0 & n \neq -1\,, \\ 2\pi\mathbf{i} & n = -1\,. \end{cases}$$

For any $n < 0$, disregard of whether $n = -1$ or $n \leq -2$, the function $f(z) = z^n$ is not defined at the origin 0. Therefore, in both situations, the circle $\gamma$ is contained in $\Omega = \mathbb{C} \setminus \{0\}$. The crucial difference is the antiderivative. We expect that if $F'(z) = f(z)$, then

$$F(z) = \begin{cases} \frac{1}{n+1} z^{n+1} & n \neq -1\,, \\ \log(z) & n = -1\,. \end{cases}$$

The antiderivative $F(z) = z^{n+1}/(n+1)$ is defined and continous on $\Omega = \mathbb{C} \setminus \{0\}$. However, $\log z$ is only a set and one must choose a branch of it. None of the branches is continuous on $\mathbb{C} \setminus \{0\}$. In order to take a continuous branch of $\log z$, we need to delete a half-line in $\mathbb{C}$ and any half-line intersects the circle with center 0.

EXERCISE 9.12. Prove that if $\gamma$ is the circle $\{\, z \in \mathbb{C} : \ |z - z_0| \,\} = R$ where $R < |z_0|$, then

$$\int_\gamma \frac{1}{z} \, dz = 0\,.$$

EXERCISE 9.13. Let $\Gamma = (\gamma_r, \gamma_\ell)$ where $\gamma_r$ is the right half semi-circle from $-R\mathbf{i}$ to $R\mathbf{i}$, while $\gamma_\ell$ is the left half from $R\mathbf{i}$ to $-R\mathbf{i}$. Show by using Theorem 9.8 (without direct calculation) that

$$\int_\Gamma \frac{1}{z} \, dz = 2\pi\mathbf{i}\,.$$