THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 7 Logarithm

7.1 Inverse to Exponential

We all know that for real variable, e^x and $\ln x$ are inverses to each other. They are important in analysis. So, we are going to discuss analogue in complex. Let us consider the situation below.

$$e^w = z = x + \mathbf{i}y \in \mathbb{C} \setminus \{0\} \quad \underset{- - - - \rightarrow}{\longleftarrow} \quad w = u + \mathbf{i}v \in \mathbb{C}\,.$$

The expression of the solid arrow is given, that is, $x = e^u \cos v$ and $y = e^u \sin v$. Analytically, to find the inverse of $w \mapsto \exp(w)$ is really re-arranging the equations and changing the subjects to u, v in terms of x, y. The first step is easy,

$$x^2 + y^2 = e^{2u}, \qquad \text{thus} \quad u = \frac{1}{2} \ln\left(x^2 + y^2\right) = \ln\sqrt{x^2+y^2} = \ln|z|\ .$$

The step about v is not that simple, we have

$$\cos v = \frac{x}{e^u} = \frac{x}{|z|}, \qquad \sin v = \frac{y}{|z|}\,.$$

First, there is not a single formula for v. Moreover, for each z, there are infinitely many solutions for v with each pair differs by a multiple of 2π. This is expected because $w \mapsto \exp(w)$ is not a 1-1 function and it should not have a single inverse function. Nevertheless, we have an expression for u and a set to describe v, namely,

$$u = \ln|z|\,, \qquad v \in \left\{ \theta \in \mathbb{R} :\ \cos\theta = \frac{x}{|z|},\ \sin\theta = \frac{y}{|z|} \right\}.$$

7.1.1 Argument

From the above, we see that given $z = x + \mathbf{i}y \in \mathbb{C} \setminus \{0\}$, an important set is associated to z,

$$\arg z \overset{\text{def}}{:=} \left\{ \theta \in \mathbb{R} :\ \cos\theta = \frac{x}{|z|},\ \sin\theta = \frac{y}{|z|} \right\}.$$

It is called the *argument* of z. Each pair of elements in $\arg z$ differs by a multiple of 2π. In addition to this set, we define the *principal argument* by the unique element in the intersection,

$$\operatorname{Arg} z \in (\arg z) \cap (-\pi, \pi]\,.$$

There are pros and cons for both $\arg(z)$ and $\operatorname{Arg}(z)$. For example, $z \mapsto \operatorname{Arg}(z)$ is a continuous function on a suitable domain. However, we have good properties such as

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)\,, \text{ etc.,}$$

which is no longer true for $\operatorname{Arg}(z)$.

7.1.2 Set of Inverses to Exponential

After introducing the set $\arg(z)$, there is a way to write the above result. Though it is not an inverse function to $w \mapsto \exp(w)$, it resembles the situation. Let $z = x + \mathbf{i}y$ and $w = u + \mathbf{i}v$ such that $z = e^w$. Then, we may write

$$\boxed{\log z \stackrel{\text{def}}{:=\!=} \ln|z| + \mathbf{i}\arg(z) \subset \mathbb{C}\,.}$$

For any $z \in \mathbb{C}\setminus\{0\}$, $\log(z)$ is a set containing complex numbers with a fixed real part and each imaginary part is picked from the set $\arg(z)$. As a consequence, if $w \in \log(z)$ then $e^w = z$. On the other hand, if $w \in \mathbb{C}$, then the set $\log(e^w) = \{w\}$. Thus, the set $\log(z)$ plays a role similar to an inverse function. In some classical book, $\log(z)$ is called a *multi-valued function*.

7.2 Branches

In many situation, it is better to have a function to work on. The most common one is the *principal logarithm*,

$$\operatorname{Log} z \stackrel{\text{def}}{:=\!=} \ln|z| + \mathbf{i}\operatorname{Arg}(z) \in \log z\,.$$

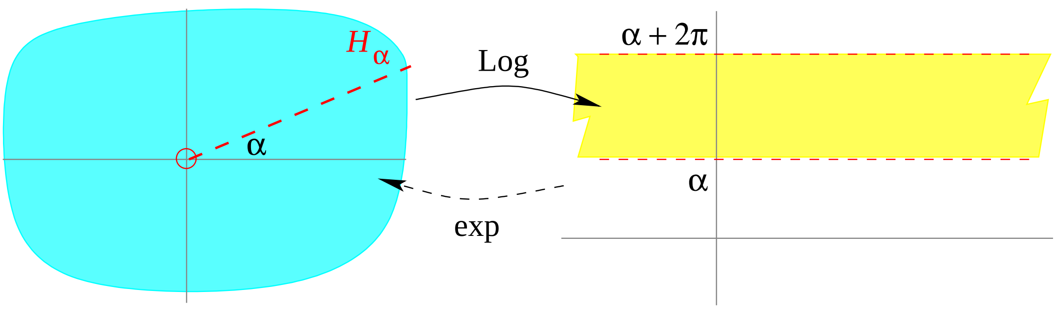
Here, $\operatorname{Log} z$ is no longer a set, but the value is a complex number in the set $\log(z)$. Apparently, it is defined for all $z \neq 0$. But then the function $z \mapsto \operatorname{Log} z$ is not continuous and it is undesirable.

Let us consider the more general case, which is sometimes more convenient, depending on the context. Let $\alpha \in \mathbb{R}$ be a fixed number. Then

$$\operatorname{Log}_\alpha z \stackrel{\text{def}}{:=\!=} \ln|z| + \mathbf{i}\operatorname{Arg}_\alpha(z), \qquad \text{where} \quad \operatorname{Arg}_\alpha(z) \in \arg(z) \cap (\alpha, \alpha + 2\pi)\,.$$

Note that the imaginary part of $\operatorname{Log}_\alpha(z)$ lies in the open interval $(\alpha, \alpha+2\pi)$ while the real part can be any real number. Then $w = \operatorname{Log}_\alpha(z)$ lies in a horizontal strip $\mathbb{R} \times (\alpha, \alpha + 2\pi)$. In such a case, for w in this strip, $\exp(w) \in \mathbb{C}\setminus H_\alpha$ where H_α is a half-line given by

$$H_\alpha \stackrel{\text{def}}{:=\!=} \left\{ r e^{\mathbf{i}\alpha} :\ r \geq 0 \right\}\,.$$



In this way, the principal logarithm $\operatorname{Log}(z)$ can be seen as $\operatorname{Log}_{-\pi}(z)$. Each $\operatorname{Log}_\alpha$ is called a *branch* of logarithm and it is continuous on the domain $\mathbb{C}\setminus H_\alpha$. In general, let $\Omega \subset \mathbb{C}\setminus\{0\}$ and $\ell :\ \Omega \to \mathbb{C}$ be a continuous function such that $\ell(z) \in \log(z)$ for all z, then ℓ is called a *branch* of logarithm on Ω. It can be proved that there is no continuous branch of $\log(z)$ on $\mathbb{C}\setminus\{0\}$.

One has to be very careful when working with branches of logarithm. For example,

$$\begin{aligned}\operatorname{Log}(-\mathbf{i})^2 &= \operatorname{Log}(-1) = \ln|-1| + \mathbf{i}\operatorname{Arg}(-1) = \mathbf{i}\pi\,,\\ 2\operatorname{Log}(-\mathbf{i}) &= 2\left[\,\ln|-\mathbf{i}| + \mathbf{i}\operatorname{Arg}(-\mathbf{i})\,\right] = 2\mathbf{i}\left(\frac{-\pi}{2}\right) = -\mathbf{i}\pi\,.\end{aligned}$$

Many equations such as $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ only are valid with an interpretation of sets, i.e., any element on the left will exist also on the right and vice versa.

7.2.1 Continuity implies Analyticity

Let $\ell : \ \Omega \subsetneq \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be a continuous branch of log on Ω. You may think of it as $\operatorname{Log}_\alpha$ for easy understanding.

Theorem 7.1. *The continuous function ℓ is automatically analytic on Ω.*

Clearly, continuity normally does not upgrade to differentiability. This is a special case for logarithm because it has an inverse relationship with exponential function. Indeed, the proof can be similarly adopted to other inverse functions.

Let $z = x + \mathbf{i}y \in \Omega$ and $w = u + \mathbf{i}v = \ell(z)$. Since, $\ell(z) \in \log(z)$, we have $\exp(\ell(z)) = e^w = z$. Thus,

$$x = e^u \cos v, \qquad y = e^u \sin v\,.$$

Exercise 7.2. Apply implicit differentiation on the above two equations, show that $u(x, y)$ and $v(x, y)$ are of C^1 and they satisfy the Cauchy-Riemann Equations on Ω.

From the result of this exercise, one sees that $z \mapsto \ell(z)$ is analytic on Ω, in which the only condition used is that u, v are continuous. In fact, you should be able to get

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}\,.$$

Recall that if a complex function f is complex differentiable, and when it is seen as a function $\mathbb{R}^2 \to \mathbb{R}^2$, its differential matrix is "almost orthogonal". That is, $[Df] \cdot [Df]^T$ is a multiple of the identity matrix. Observe from the above situation of $z \mapsto \log(z)$ and $w \mapsto e^w$, can you answer the following?

Exercise 7.3. Let $g : \ \Omega \to \mathbb{C}$ be an inverse function of an analytic function f such that it is continuous on Ω. Is it true that g will be automatically analytic?