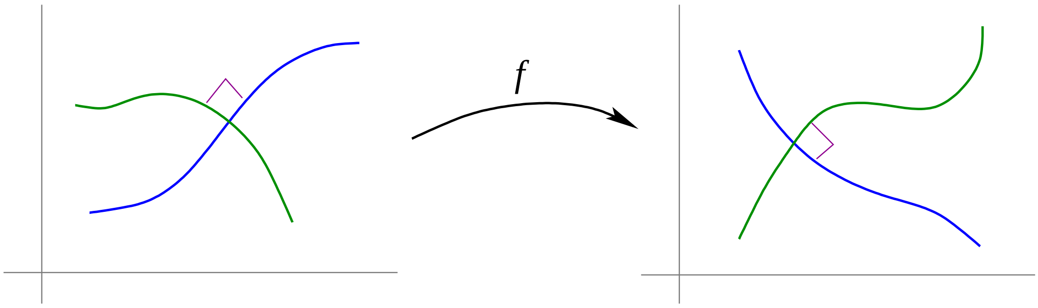
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 3 Complex Differentiability

3.1 Definition and ...

As we have always mentioned, our object of study is

$$f: \Omega \subset \mathbb{C} \to \mathbb{C}.$$

We have given some examples of such functions. In these examples, the functions exhibit a good property that they take initially perpendicular curves to perpendicular image curves.



Everybody who has some experiences in dealing with functions knows that this good property is not guaranteed by continuous functions. In other words, the examples that we showed are *better* than being continuous. ***What*** is the natural condition which is ***next to continuity***? Yes, that is "differentiability". For a complex function, the following concept is natural.

Definition 3.1. A function $f: \Omega \subset \mathbb{C} \to \mathbb{C}$ is *complex differentiable* at $z_0 \in \Omega$ if

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\zeta\to 0} \frac{f(z_0+\zeta)-f(z_0)}{\zeta} \quad \text{exists.}$$

The limit is denoted by $f'(z_0)$, which is called the *complex derivative* of f at z_0.

This definition is formally exactly the same as the derivative of a $\mathbb{R} \to \mathbb{R}$ function. In other words, if all the symbols f, z, ζ are seen as real numbers, then this is merely the well-known definition (or first principle) of derivative. However, when these symbols are seen as complex numbers, the division in the fraction is complex. Moreover, the limit $z \to z_0$ occurs in a 2-dimensional plane. It turns out these are crucial differences between real differentiability versus complex differentiability. In order to see this, it is better to write an equivalent definition.

Exercise 3.2. Show that $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = p + \mathbf{i}q$ for $p, q \in \mathbb{R}$ if and only if

$$f(z) = f(z_0) + (p+\mathbf{i}q)(z-z_0) + \varepsilon(z) \qquad \text{where} \quad \lim_{z\to z_0} \frac{|\varepsilon(z)|}{|z-z_0|} = 0.$$

Note that in the above, the limit can also be $\lim_{z\to z_0} \varepsilon(z)/(z-z_0)$. Also, to prove the equivalence, it does not matter whether the symbols are real or complex. The whole proof is only formally using properties of addition, multiplication, and limits.

3.2 See Things on the plane

As we often write $f(z) = f(x + \mathbf{i}y) = u(x, y) + \mathbf{i}v(x, y)$. We may see the function f as

$$(x, y) \in \Omega \subset \mathbb{R}^2 \to (u, v) \in \mathbb{R}^2 .$$

What is the definition of *differentiability* of this function at (x_0, y_0) where $z_0 = x_0 + \mathbf{i}y_0$?

There may be several ways to express it, but let us use the following matrix version,

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} \varepsilon_1(x,y) \\ \varepsilon_2(x,y) \end{pmatrix} ,$$

where $\dfrac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \longrightarrow 0$ as $(x, y) \to (x_0, y_0)$ for a suitable 2×2 matrix.

In fact, the above is the uniform way of writing definition for differentiability in all situations. Differentiability is always a "measurement" of how the function differs from a linear function by an error under suitable control. At this moment, we do not care the exact information of the 2×2 matrix, which is called the *differential* of f at (x_0, y_0). It is usually denoted $Df_{(x_0,y_0)}$. Some of you may know that it is made up of partial derivatives.

Now, recall in the exercise above, the equation for complex differentiability at z_0 is of the form

$$f(z) = f(z_0) + (p + \mathbf{i}q)(z - z_0) + \varepsilon(z) \qquad \text{where} \quad \lim_{z \to z_0} \frac{|\varepsilon(z)|}{|z - z_0|} = 0 .$$

Let us compare the real part and imaginary of this equation and write it into matrix. Then

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} \varepsilon_1(x,y) \\ \varepsilon_2(x,y) \end{pmatrix} .$$

From this, we see that if f is complex differentiable at z_0, then the function $(x, y) \mapsto (u, v)$ must be differentiable ***with an additional property*** on the entries of it.

Together with the fact that the differential matrix is given by

$$Df_{(x_0,y_0)} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} ,$$

the additional condition becomes the famous Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at } (x_0, y_0).$$

Let us summarize.

THEOREM 3.3. *If $f = u + \mathbf{i}v : \ \Omega \subset \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_0 = x_0 + \mathbf{i}y_0$, then the function $(x, y) \mapsto (u, v)$ is differentiable at (x_0, y_0) and its partial derivatives satisfy the Cauchy-Riemann Equations.*

It is obvious that the converse of the theorem is also true; but merely existence of partial derivatives satisfying Cauchy-Riemann Equations is not enough.

Theorem 3.3. *A function f is complex differentiable at z_0 if and only if the function $(x, y) \mapsto (u, v)$ is differentiable at (x_0, y_0) and the Cauchy-Riemann Equations are satisfied at (x_0, y_0), i.e.,*

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} \qquad \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} = -\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} .$$

Note that the Cauchy-Riemann Equations are equivalent to that the differential matrix of f is of special form, i.e.,

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}_{(x_0,y_0)} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}_{(x_0,y_0)} .$$

Some books derive the Cauchy-Riemann Equation by another method. Recall that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \qquad \text{if the limit exists.}$$

Exercise 3.4. By taking the limit along the x-axis and y-axis respectively, show that

$$f'(z_0) = \frac{\partial u}{\partial x} + \mathbf{i}\frac{\mathrm{d}v}{\mathrm{d}x} = -\mathbf{i}\left(\frac{\partial u}{\partial y} + \mathbf{i}\frac{\partial v}{\partial y}\right), \qquad \text{at } z_0 = x_0 + \mathbf{i}y_0.$$

This exercise also establishes the existence of partial derivatives and Cauchy-Riemann Equation. But, note that this proof is not "reversible".

3.2.1 In Polar Coordinates

Very often, it would be convenient to work with polar coordinates. To derive the Cauchy-Riemann Equations in polar coordinates, it would be easier to work with the matrix version and the composition diagram below.

$$\begin{tikzcd}
(x, y) \arrow[dr] \arrow[d, leftrightarrow] & \\
(r, \theta) \arrow[r] & (u, v)
\end{tikzcd}$$

According to the multivariable Chain Rule, at wherever the function is differentiable, we have

$$\begin{aligned}
\begin{bmatrix} u_r & u_\theta \\ v_r & v_\theta \end{bmatrix} &= \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \cdot \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} \qquad \text{where} \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta , \end{cases} \\
&= \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} .
\end{aligned}$$

Together with the Cauchy-Riemann Equations, we have, for instance,

$$v_r = -u_y\cos\theta + u_x\sin\theta \quad \text{and} \quad u_\theta = r\left(-u_x\sin\theta + u_y\cos\theta\right) .$$

Thus, $u_\theta = -rv_r$ and, similarly $v_\theta = ru_r$. These are the Cauchy-Riemann Equations in polar coordinates. There may be other change of variables which leads to a different version of the Equations. The principle of its derivation is mainly the Chain Rule.